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LOCAL CONTRACTIBILITY OF THE GROUP OF HOMEOMORPHISMS OF A MANIFOLD

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In this paper the group of homeomorphisms of an arbitrary topological manifold is considered, with either the compact-open, uniform (relative to a fixed metric), or majorant topology. In the latter topology, a basis of neighborhoods of the identity is given by the strictly positive functions on the manifold, a homeomorphism being in the neighborhood determined by such a function if it moves each point less than the value of this function at the point. The main result of the paper is the proof of the local contractibility of the group of homeomorphisms in the majorant topology. Examples are easily constructed to show that this assertion is false for the other two topologies for open manifolds. In the case of a compact manifold the three topologies coincide. In conclusion a number of corollaries are given; for example, if a homeomorphism of a manifold can be approximated by stable homeomorphisms then it is itself stable.

§1. Statement of the problem and formulation of the results

1.1. In this paper the local contractibility of the group of homeomorphisms of a metrizable manifold M with a fixed metric $\rho(x, y)$ is studied. We denote this group by $\mathfrak{H} = \mathfrak{H}(M)$ and endow it with one of the three topologies: compact-open, uniform, majorant. The group $\mathfrak{H}(M)$ when so topologized will be denoted by $\mathfrak{H}_c(M)$, $\mathfrak{H}_u(M)$, or $\mathfrak{H}_m(M)$ respectively, or by $\mathfrak{H}_\tau(M)$ if the topology is unspecified. A basis of neighborhoods of the identity $e = e(M)$ (the identity mapping) is given in $\mathfrak{H}_c(M)$ by the pairs (K, ϵ), where $\epsilon > 0$ and K is a compact subset of M; the neighborhood determined by the pair (K, ϵ) is denoted by $\Omega_{K,\epsilon}(e)$, and consists of all homeomorphisms $h\colon M \to M$ such that $\rho(x, hx) < \epsilon$ for $x \in K$. A basis of neighborhoods of e in $\mathfrak{H}_u(M)$ is given by the numbers $\epsilon > 0$; the neighborhood determined by ϵ is denoted by $\Omega_\epsilon(e)$, and consists of all homeomorphisms h such that $\rho(x, hx) < \epsilon$ for all $x \in M$. In $\mathfrak{H}_m(M)$ a basis of neighborhoods of the identity is given by the continuous strictly positive functions on M, which we shall call *majorants*: the neighborhood determined by the majorant $f\colon M \to (0, \infty)$ is denoted by $\Omega_f(e)$, and consists of all h such that $\rho(x, hx) < fx$ for all $x \in M$.

The main result of the paper is the proof of the local contractibility of $\mathfrak{H}_m(M)$ for all manifolds and of $\mathfrak{H}_c(M)$ for compact manifolds. However the exact sense of, at any rate, the first assertion requires clarification. This is the main purpose of the first section. A complete formulation of the results is given in subsections 13, 14, 21, and 22 of the present section and in §5. An outline of the proof is given in subsection 1.26.

We denote by $[X]$, $\operatorname{Int} X$, and $\operatorname{Fr} X$ the closure, interior, and frontier of the set X in the manifold M. By ∂M and $\operatorname{Int} M$ we denote the boundary and the interior of the manifold; $O_\epsilon(X)$ is the ϵ-neighborhood of the set X in M. We denote the empty set by Λ.

1.2. It is easy to see that for all three values of τ the group $\mathfrak{H}_\tau(M)$ is a topological group, and a topological group of transformations of M. One must verify the continuity of the three mappings

1) $\mathfrak{H}_\tau(M) \times \mathfrak{H}_\tau(M) \to \mathfrak{H}_\tau M$: $(h, h') \to hh'$,

2) $\mathfrak{H}_\tau(M) \to \mathfrak{H}_\tau(M)$: $h \to h^{-1}$,

3) $\mathfrak{H}_\tau(M) \times M \to M$: $(h, x) \to hx$.

The continuity of the first and third mappings is almost obvious, and that of the second is clear for $\tau = u$ and $\tau = m$. Let us verify the continuity of the second mapping for $\tau = c$ at the point e (cf. [1]).

Let $\Omega_{K,\epsilon}(e)$ be a given neighborhood of e in $\mathfrak{H}_c(M)$. Let K' be a compact subset of M containing K in its interior, and ϵ' a positive number less than ϵ, so small that the image of K' under any (homeomorphic) ϵ'-shift of K' in M contains K in its interior. The existence of such an ϵ' follows from homology considerations: it is sufficient to require that on $\operatorname{Fr} K'$ an ϵ'-shift be homotopic to the identity mapping outside K. Then it is clear that if $h \in \Omega_{K',\epsilon'}(e)$ then $h^{-1} \in \Omega_{K,\epsilon}(e)$.

1.3. Clearly, for compact manifolds the three topologies coincide. The topology $\tau = u$ depends in general on the metric in M, but occupies an intermediate position between the topologies $\tau = c$ and $\tau = m$ (which are independent of the metric), in the sense that the identity mappings

$$\mathfrak{H}_m(M) \to \mathfrak{H}_u(M) \to \mathfrak{H}_c(M)$$

are continuous (they are contractions).

1.4. Definition 1. An *isotopy* of the manifold M means a layer homeomorphism of $M \times [0, 1]$ onto itself.

Being a layer homeomorphism means that the isotopy Φ: $M \times [0, 1] \to M \times [0, 1]$ determines homeomorphisms $(\Phi)_t$: $M \to M$ such that $\Phi(x, t) = ((\Phi)_t x, t)$ for each point $(x, t) \in M \times [0, 1]$. We shall say that Φ joins the homeomorphisms $(\Phi)_0$ and $(\Phi)_1$, or that it takes $(\Phi)_0$ into $(\Phi)_1$.

(In order to distinguish the lower index which gives the value of the isotopy parameter t, we shall always attach it to a parenthesis, so that it will always be the outermost index.)

1.5. Remark 1. It is sufficient to require that Φ be a one-to-one continuous layer mapping of $M \times [0, 1]$ onto itself, since from the theorem on invariance of domains, applied to the manifold $M \times [0, 1]$, follows the continuity of the mapping inverse to Φ (cf. [2]).

1.6. Our definition of isotopy does not depend on the topology in the group of homeomorphisms $\mathfrak{H}(M)$. It might seem that for the study of homotopic properties of these groups, for example local contractibility, the definition of isotopies as paths in $\mathfrak{H}_\tau(M)$ would be more natural. As is well known, the two definitions are equivalent for $\tau = c$ (see [3], [4]). This is not so in the other two cases.

Namely, the group $\mathfrak{H}_m(M)$ for a noncompact manifold possesses no countable basis, and moreover if $h_i \to h$ is a convergent sequence of homeomorphisms then from some index on they all coincide with their limit outside some compact set, the same for all of them. Indeed, otherwise there exists a sequence of pairwise distinct homeomorphisms converging to e and a sequence of points $\{x_i\}$ with no convergent subsequence, such that $h_i x_i \neq x_i$. By this property of $\{x_i\}$ we can find a majorant f with the property that $fx_i < \rho(x_i, h x_i)$, and then all the h_i lie outside $\Omega_f(e)$, so that it is not true that $\lim h_i = e$. Hence a path in $\mathfrak{H}_m(M)$ can only join homeomorphisms that coincide outside some compact set. Thus in the general case $\mathfrak{H}_m(M)$ is not even locally arcwise connected when M is noncompact.

As regards the topology $\tau = u$, convergence in the sense of this topology is uniform convergence, whence it is easily deduced that if J: $[0, 1] \to \mathfrak{H}_u(M)$ is a continuous mapping of an interval then there must exist a function $\chi(t, t')$, $t, t' \in [0, 1]$, defined on the unit square, such that $\chi(t, t') = \chi(t', t)$, $\chi(t, t) = 0$ and $\chi(t, t') > 0$ for $t \neq t'$, and $\rho(J(t)x, J(t')x) \leq \chi(t, t')$ for $\chi \in M$. Again, this condition is too restrictive.

1.7. We consider the set of all isotopies of M as a subgroup $\mathfrak{I}(M)$ of the group $\mathfrak{H}(M \times [0, 1])$, and we topologize it as a subspace of $\mathfrak{H}_\tau(M \times [0, 1])$. The group $\mathfrak{I}(M)$ with this topology is denoted by $\mathfrak{I}_\tau(M)$. (The direct product metric is taken in $M \times [0, 1]$.)

1.8. We introduce some further concepts and notation. We denote the unit of the group $\mathfrak{I}_\tau(M)$ (the identity isotopy) by E, or occasionally, if necessary, by $\mathrm{E}(M)$. The isotopy inverse to Φ is denoted by Φ^{-1}: $(\Phi^{-1})_t = ((\Phi)_t)^{-1}$. If for the isotopy Φ we have $(\Phi)_t x = hx$ for all $t \in [0, 1]$ on the set $X \subset M$, then we shall say that Φ is identically equal to h on X, and write $\Phi \equiv h$ on X.

The support of an isotopy Φ is a set $S = S(\Phi)$ such that all the $(\Phi)_t$ coincide outside S (but are not necessarily the identity!).

As well as the *product* $\Phi\Psi$ of isotopies Ψ and Φ (where $(\Phi\Psi)_t = (\Phi)_t(\Psi)_t$), we shall also consider their *composition* $\Psi \circ \Phi$, which is defined, provided only that $(\Psi)_1 = (\Phi)_0$, as the composition of homotopies:

$$(\Psi \circ \Phi)_t = (\Psi)_{2t}, \text{ for } 0 \leq t \leq 1/2, \quad (\Psi \circ \Phi)_t = (\Phi)_{2t-1}, \text{ for } 1/2 \leq t \leq 1.$$

In addition, we shall need to consider the product Φh of an isotopy and a homeomorphism, which we understand as the product of Φ and the isotopy $\Psi \equiv h$.

Finally we shall consider infinite compositions. If a sequence of isotopies $\Phi_1, \Phi_2, \cdots$ is given then their infinite composition is defined only if $(\Phi_{i+1})_0 = (\Phi_i)_1$, $i \geq 1$, and it is the layer homeomorphism Φ: $M \times [0, 1) \to M \times [0, 1)$, where $(\Phi)_t = (\Phi_i)_{(i+1)(ti-i+1)}$ for $t \in [(i-1)/i, i/(i+1)]$. We shall say that the sequence $\{\Phi_i\}$ converges if its infinite composition is defined and extends to a continuous mapping of $M \times [0, 1]$ onto itself. For this it is obviously necessary and sufficient that there should be a continuous mapping $(\Phi)_1$: $M \to M$, where $(\Phi)_1 = \lim (\Phi_i)_{t_i}$, in the sense $\tau = c$, for any sequence t_i, where all the $t_i \in [0, 1]$. We call $(\Phi)_1$ the *limit mapping*, and Φ, completed by $(\Phi)_1$ in the way described, the *limit pseudoisotopy*. If $(\Phi)_1$ is a homeomorphism then Φ is an isotopy, by Remark 1 (see 1.5), and we shall also call it the limit isotopy.

1.9. Remark 2. For any neighborhood $\Omega(e)$ in the group $\mathfrak{H}_\tau(M)$ there is a neighborhood $\mathfrak{O}(\mathrm{E})$ such that for $\Phi \in \mathfrak{O}(\mathrm{E})$ all the homeomorphisms $(\Phi)_t$ lie in $\Omega(e)$.

In the case $\tau = c$, for a neighborhood $\Omega_{K,\epsilon}(e)$ in $\mathfrak{H}_c(M)$ we take as the required $\mathfrak{O}(\mathrm{E})$ the neighborhood $\Omega_{K\times[0,1],\epsilon}(\mathrm{E}) \cap \mathfrak{I}_c(M)$ in $\mathfrak{I}_c(M)$. In the case $\tau = u$, for the neighborhood $\Omega_\epsilon(e)$ we take the neighborhood $\Omega_\epsilon(\mathrm{E}) \cap \mathfrak{I}_u(M)$ in $\mathfrak{I}_u(M)$. In the case $\tau = M$, for the neighborhood $\Omega_f(e)$ we take the neighborhood $\Omega_{\bar{f}}(\mathrm{E}) \cap \mathfrak{I}_m(M)$, where $\bar{f}$ is the majorant equal to $f(x)$ for all points (x, t), $t \in [0, 1]$.

We note also that for each majorant f on $M \times [0, 1]$ there is a majorant f' on M such that $f(x, t) < f'x$ for all $x \in M$.

1.10. We now pass to definitions concerning homotopies in $\mathfrak{H}_\tau(M)$.

Definition 2. *A subset* $\mathrm{A} \subset \mathfrak{H}_\tau$ *deforms on* $\mathrm{B} \subset \mathfrak{H}_\tau$ *into* $\Gamma \subset \mathfrak{H}_\tau (\mathrm{A} \cup \Gamma \subset \mathrm{B})$ if there is a continuous mapping Υ: $\mathrm{A} \to \mathfrak{I}_\tau$ such that for $h \in \mathrm{A}$ we have

1) $(\Upsilon(h))_0 = h$;
2) $(\Upsilon(h))_t \in \mathrm{B}$ for $t \in [0,1]$;
3) $(\Upsilon(h))_1 \in \Gamma$.

We now introduce the main concept of this paper.

Definition 3. The group $\mathfrak{H}_\tau(M)$ is called *locally contractible* if there is a neighborhood of e in $\mathfrak{H}_\tau(M)$ which deforms on $\mathfrak{H}_\tau(M)$ into e.

1.11. Remark 3. In this definition of local contractibility we may always assume that $\Upsilon(e) = \mathrm{E}$, since any contraction $\Upsilon(h)$ can be replaced by the contraction $\Upsilon'(h) = (\Upsilon(e))^{-1}\Upsilon(h)$, having this property. Further, an arbitrary neighborhood $\Omega(e)$ may be taken as B, provided that a sufficiently small neighborhood of e is taken as $\mathrm{A} = \Omega'(e)$. For by Remark 2 there is a neighborhood $\mathfrak{O}(\mathrm{E}) \subset \mathfrak{J}_\tau$ such that for each isotopy $\Phi \in \mathfrak{O}$ all the homeomorphisms $(\Phi)_t \in \Omega(e)$. Since Υ is continuous, there is a *neighborhood* $\Omega'(e)$ *such that* $\Upsilon(\Omega') \subset \mathfrak{O}$. Thus *if* $\mathfrak{H}_\tau$ *is* locally contractible *in the sense of* Definition 3 then for a given neighborhood $\Omega(e)$ there is a neighborhood $\Omega'(e)$ which deforms into e in $\Omega(e)$. This agrees with the usual definition.

On the other hand, one can weaken the definition and require only that some open subset of $\mathfrak{H}_\tau$ deforms in $\mathfrak{H}_\tau$ into some point of $\mathfrak{H}_\tau$.

1.12. We make the following further two definitions.

Definition 4. A subset $\mathrm{A} \subset \mathfrak{H}_\tau(M)$ is called *contractible* if it contracts in itself to a point.

Definition 5. The group $\mathfrak{H}_\tau(M)$ *contracts locally into its subset* A, containing e, if there is a neighborhood $\Omega(e)$ which contracts in $\mathfrak{H}_\tau$ into A with e fixed.

Finally, we denote by $\Delta(X)$ the subgroup of homeomorphisms that are fixed on the subset $X \subset M$.

1.13. We now formulate the main result of the paper.

Fundamental Theorem. *For any metrizable manifold M the group $\mathfrak{H}_m(M)$ is locally contractible.*

1.14. In fact we shall prove an essentially stronger result (see Proposition (B) in 1.22). In the compact case the three topologies coincide, as we have already said, and so as a corollary of the fundamental theorem we have

Theorem 1. *If the manifold M is compact then the group $\mathfrak{H}_c(M)$ is locally contractible.*

1.15. We recall that the local contractibility of the group $\mathfrak{H}_c$ was hitherto known only for two-dimensional compact manifolds, and also its local p-connectedness for all p for compact three-dimensional manifolds (results of Hamstrom [5], [6]). In addition Kister [8], by modifying the well-known argument of Alexander [7], proved the local contractibility of $\mathfrak{H}_u(R^n)$ for Euclidean space R^n with its usual metric.

1.16. If a manifold has a boundary, then by applying the Fundamental Theorem we can find a contraction $\Upsilon : \Omega(e) \to \mathfrak{J}_m(M)$ of some neighborhood $\Omega(e)$ into e in $\mathfrak{H}_m(M)$, such that if $h \in \Omega(e) \cap \Delta(\partial M)$ then $\Upsilon(h) = \mathrm{E}$ on ∂M.

For let Ω^∂ be a neighborhood of $e^\partial = e(\partial M)$ in $\mathfrak{H}_m(\partial M)$ for which, by the Fundamental Theorem, there exists a contraction $\Upsilon^\partial : \Omega^\partial \to \mathfrak{J}_m(\partial M)$ into e^∂, where by 1.11 we may assume that $\Upsilon^\partial(e^\partial) = e^\partial$. Now let $\widetilde{\Upsilon} : \Omega(e) \to \mathfrak{J}_m(M)$ be a given contraction of some neighborhood $\Omega(e)$ into $e(M)$ such that if $h \in \Omega(e(M))$ then $h^\partial = h|_{\partial M} \in \Omega^\partial$. For each homeomorphism h^∂ which is the restriction of a homeomorphism h in Ω, and is considered as such, it induces an isotopy $\widetilde{\Upsilon}^\partial(h)$. Thus for each $h \in \Omega$ we have two isotopies on ∂M for $h^{\partial}|_{\partial M}$, namely $\Upsilon^{\partial}(h^{\partial})$ and $\widetilde{\Upsilon}^{\partial}(h)$. This gives for each $h \in \Omega$ a layer homeomorphism

$$\Upsilon^{\partial}_{t,t'}(h): \partial M \times [0,1] \times [0,1] \to \partial M \times [0,1] \times [0,1],$$

depending continuously on $h: \Upsilon^{\partial}_{t,t'}(h)x = (\Upsilon^{\partial}((\widetilde{\Upsilon}^{\partial}(h))_t)_{t'})x$, $x \in \partial M$, where $\Upsilon^{\partial}_{0,0}(h) = h^{\partial}$ and $\Upsilon^{\partial}_{1,t'}(h) = \Upsilon^{\partial}_{t,1}(h) = e^{\partial}$. Constructing in the square $[0, 1] \times [0, 1]$ the segments joining the point $(0, 0)$ with the points of the sides $1 \times [0, 1]$ and $[0, 1] \times 1$, we obtain a family of isotopies $\Upsilon^{\partial}_s(h)$ of the boundary of M depending on $h (\tau = m)$ and on $s \in [0, 1]$ $(\tau = c)$, such that $\Upsilon^{\partial}_0(h) = \widetilde{\Upsilon}^{\partial}(h)$ and $\Upsilon^{\partial}_1(h) = \Upsilon^{\partial}(h^{\partial})$. We recall that if $h^{\partial} = e^{\partial}$ then $\Upsilon^{\partial}(h^{\partial}) = \mathrm{E}(\partial M)$.

According to Brown [9] there is a homeomorphism G: $\partial M \times [0, 1] \approx Q$, where Q is a closed neighborhood of the boundary. Let $G(\partial M \times 0) = \partial M$. Let $Q_s = G(\partial M \times s)$, and let P_s: $\partial M \times s \to \partial M$ be the homeomorphism induced by the projection of the direct product $\partial M \times [0, 1] \to \partial M$. Now construct a contraction $\Upsilon: \Omega \to \mathfrak{I}_m(M)$ as follows: for $h \in \Omega$ the isotopy $\Upsilon(h)$ is equal to $\widetilde{\Upsilon}(h)$ outside Q and is equal to $\widetilde{\Upsilon}(h) \cdot G \cdot P_s^{-1} \cdot (\widetilde{\Upsilon}^{\partial}(h))^{-1} \cdot \Upsilon^{\partial}_{1-s}(h) \cdot P_s \cdot G^{-1}$ on Q_s. One verifies directly that on $Q_0 = \partial M$ the isotopy $\Upsilon(h)$ is equal to $\Upsilon^{\partial}(h^{\partial})$ and so it is equal to $\mathrm{E}(\partial M)$ if $h^{\partial} = e^{\partial}$. At the same time $\Upsilon(h)$ is equal to $\widetilde{\Upsilon}(h)$, on Q_1, and so the two definitions are equivalent.

1.17. We can derive the Fundamental Theorem for manifolds with boundary by applying it to manifolds without boundary, but in the somewhat stronger form:

If D is a closed subset of M and $O(\mathrm{Fr}\, D)$ is a neighborhood of the boundary of D, then the subgroup $\Delta(D) \subset \mathfrak{H}_{\tau}(M)$ is contractible into e in $\Delta(D \setminus O(\mathrm{Fr}\, D))$.

(We note that when D is empty this assertion becomes the Fundamental Theorem.) For the proof we construct, as in 1.16, a homeomorphism G: $\partial M \times [0, 3] \approx Q$, where Q is a closed neighborhood of ∂M in M, with $G(\partial M \times 0) = \partial M$. Again let $Q_s = G(\partial M \times s)$ and put $Q[t_1, t_2] = G(\partial M \times [t_1, t_2])$. Let q_t: $Q[t, 2] \approx Q[0, 2]$ be the homeomorphism induced by the linear homeomorphism $[t, 2] \to [0, 2]$ with the point 2 fixed, where $t \in [0, 1]$. Extend q_t identically onto $M \setminus Q[0, 2]$. Now apply the Fundamental Theorem to ∂M and construct a contraction $\Upsilon^{\partial}: \Omega^{\partial} \to \mathfrak{I}_m(\partial M)$ for some neighborhood $\Omega^{\partial} = \Omega(e^{\partial})$. Now let $\overline{\Omega}$ be a neighborhood of $e(M)$ so small that for $h \in \overline{\Omega}$ we have

1) $h^{\partial} = h|_{\partial M} \in \Omega^{\partial}$,

2) $hQ \supset Q[0, 2]$.

Now construct an isotopy $\Upsilon_1(h)$ for $h \in \overline{\Omega}$ as follows:

$$\begin{aligned} &(\Upsilon_1(h))_t = h && \text{on } Q, \\ &(\Upsilon_1(h))_t = q_t^{-1} h q_t && \text{on } Q[t, 3], \\ &(\Upsilon_1(h))_t = G P_s^{-1} h^{\partial} P_s G^{-1} && \text{on } Q_s,\ s \in [0, t], \end{aligned}$$

where P_s: $\partial M \times s \to \partial M$ is induced by the projection of the direct product. These definitions are consistent, since $q_t = e$ on Q_3, and $hQ_3 \subset M \setminus Q[0, 2]$ by condition 2), and so $q_t = e$ on hQ_3. Thus on Q_3 the first and second definitions give $(\Upsilon_1(h))_t = h$, $t \in [0, 1]$. Further $q_t = P_s G^{-1}$ on Q_t, and so the second and third definitions give the same on Q_t. Thus we obtain an isotopy. It is clear that $\Upsilon_1(h)$ depends continuously on h, with $\Upsilon_1(e) = \mathrm{E}$. The isotopy $\Upsilon_1(h)$ takes h into the homeomorphism $(\Upsilon_1(h))_1$, which is equal to $G P_s^{-1} h^{\partial} P_s G^{-1}$ on Q_s, where $s \in [0, 1]$.

Now let $\Upsilon_2(h)$ be the isotopy defined as follows:

$$\begin{aligned}
(\Upsilon_2(h))_t &= (\Upsilon_1(h))_1 && \text{on } M\setminus Q[0,1],\\
(\Upsilon_2(h))_t &= GP_s^{-1}(\Upsilon^{\partial}(h^{\partial}))_t P_s G^{-1} && \text{on } Q[0, {}^1/_2],\\
(\Upsilon_2(h))_t &= GP_s^{-1}(\Upsilon^{\partial}(h^{\partial}))_{2t(1-s)} P_s G^{-1} && \text{on } Q_s,\ s\in[{}^1/_2, 1].
\end{aligned}$$

It is again easy to verify that the three definitions are consistent. Namely, the third definition coincides for $s = 1$ with the first, and for $s = \frac{1}{2}$ with the second. Again $\Upsilon_2(h)$ depends continuously on h, and $\Upsilon_2(h) = \mathrm{E}$, if $h = e$. The composition of the isotopies $\Upsilon_1(h)$ and $\Upsilon_2(h)$ is defined, and takes h into a homeomorphism which is the identity on $Q[0, \frac{1}{2}]$. Now we apply the Fundamental Theorem, with the refinement mentioned at the beginning of this subsection, to $\operatorname{Int} M$, with $Q[0, \frac{1}{2}]$ taken as D and $Q[\frac{1}{4}, \frac{3}{4}]$, say, as $O(\operatorname{Fr} D)$. For some neighborhood $\tilde{\Omega}(e(\operatorname{Int} M))$ we have a contraction $\Upsilon_3 : \tilde{\Omega}(e) \cap \Delta(Q[0, {}^1/_2]) \to \mathfrak{J}_\tau(\operatorname{Int} M)$ in $\Delta(Q[0, \frac{1}{4}])$, which can be extended to the boundary as the identity. It is obvious that for $h = e$ the composition $\Upsilon_1(e) \circ \Upsilon_2(e) = \mathrm{E}$. Now there exists a neighborhood $\Omega(e) \subset \tilde{\Omega}$ so small that if $h \in \Omega$ then the composition $\Upsilon_2(h) \circ \Upsilon_1(h)$ takes h into a homeomorphism lying in $\tilde{\Omega}$. So the composition $\Upsilon_1(h) \circ \Upsilon_2(h) \circ \Upsilon_3(h)$, is defined, depends continuously on h, and takes h into the homeomorphism which is the identity on the whole manifold.

Thus we may validly assume that our given manifold is without boundary.

We note if we consider only homeomorphisms which are the identity on some closed set D, then by a slight strengthening of this reasoning we could show that the resulting improvement of the theorem is again valid for manifolds with boundary if it is valid for manifolds without boundary.

1.18. In the case of a noncompact manifold M one may speak of the local contractibility of $\mathfrak{H}_c(M)$ or $\mathfrak{H}_u(M)$ only if the topology of the manifold is sufficiently simple at infinity. For example let $M = \bigcup_{i=0}^{\infty} T_i$, where each T_i is a two-dimensional torus with two holes and $T_i \cap T_{i+1} = \gamma_{i+1}$, where γ_i and γ_{i+1} are the boundaries of the holes in T_i for $i \geq 0$, the diameter of T_i being further assumed less than $1/i$. Define homeomorphisms h_i: $M \to M$ as follows: if H_i is a cylindrical neighborhood of γ_i in T_i then $h_i = e$ on $M\setminus H_i$ and h_i is a twist through 2π on H_i. Then the curve which is the product of the meridians of the tori T_{i-1} and T_i is $\pi_i M$ is not homotopic to its image under the homeomorphism h_i, which is the product $m_{i-1} l_{i-1} m_i l_{i-1}^{-1}$, where m_i is the meridian of T_i and l_i is the class of γ_i. (We remark that the group $\pi_i M$ is freely generated by the meridians m_i and parallels p_i of the tori and also l_0, with $l_i = l_0 \Pi_{i'=1}^{i} [p_{i'}, m_{i'}]$.) Thus none of the homeomorphisms h_i is even homotopic to e, although it is clear that $\lim h_i = e$ in the topologies $\tau = c$ or $\tau = u$, and so the groups $\mathfrak{H}_c(M)$ and $\mathfrak{H}_u(M)$ are not even semilocally linearly connected.

1.19. On the other hand, if an open manifold M has finitely generated homology and is simply connected at infinity, then in the case when $\dim M \geq 6$ [10], or in the case of a three-dimensional irreducible M [11], there is a compact manifold whose interior is homeomorphic to M. (If the condition of simply-connectedness at infinity is omitted then this is not in general true [2].) Thus although in the case of an open manifold we cannot give a definitive answer when $\tau = c$ or $\tau = u$, we nevertheless see that with a high degree of generality we can restrict ourselves to the case when M is the interior of a compact manifold, and we now consider this case.

1.20. Let $M = \operatorname{Int} N$, where N is a compact manifold with nonempty boundary. Suppose further that the metric of M is induced by that of N, in the case $\tau = u$. As before, let G: $\partial N \times [0, 1] \approx Q$ be a homeomorphism on a closed neighborhood Q of the boundary of N and let $G(\partial N \times 0) = \partial N$. We note that $K = [N\setminus Q]$ is a compact subset of $M = \operatorname{Int} N$. Consider the subgroup $\Delta(K) \subset \mathfrak{H}_\tau(M)$, where

$\tau = c$ or $\tau = u$. For $t \in (0, 1]$ define a homeomorphism $q_t\colon Q \longrightarrow Q$ by the formula $q_t(G(x, s)) = G(x, s^{1/t})$, where $x \in \partial N$ and $s \in [0, 1]$. If $h \in \Delta(K)$ let

$$\begin{aligned} &(\Phi(h))_t = q_t^{-1} h q_t && \text{on } Q \diagdown \partial N \quad \text{for } t \in [0, 1], \\ &(\Phi(h))_t = e && \text{on } N \diagdown G \quad \text{for } t \in [0, 1], \\ &(\Phi(h))_0 = e && \text{on the whole of } M. \end{aligned}$$

This defines an isotopy $\Phi(h)$ on M, depending continuously on h. Thus $\Delta(K)$ is a contractible subgroup for $\tau = c$ or $\tau = u$. This reasoning shows that in the present case the local contractibility of $\mathfrak{H}_c(M)$ or $\mathfrak{H}_u(M)$ is a consequence of the following more general assertion.

1.21. **Proposition** (A). *For each compact subset K of the manifold M and for all values $\tau = c$, u, or m, the group $\mathfrak{H}_\tau(M)$ is locally contractible in $\Delta_\tau(K)$.*

As was shown in 1.20, this implies

Theorem 2. *If the manifold M is the interior of a compact manifold N, then $\mathfrak{H}_c(M)$, and also $\mathfrak{H}_u(M)$ if the metric of M is induced by that of N, is locally contractible.*

In fact we shall prove the following assertion, essentially more general than (A), from which we shall also derive the Fundamental Theorem.

1.22. **Proposition** (B). *If C and D are closed subsets of the open manifold M, then for each neighborhood $O = O(C)$ and each neighborhood $O' = O'(\mathrm{Fr}\, D \cap C)$ there exists a strictly positive function f on O, such that for each homeomorphism h for which $\rho(x, hx) < fx$, $x \in O$, and which is the identity on D, there is an isotipy $\Upsilon(h)$ such that*

1(B). $\Upsilon(h)$ *depends continuously on* h;
2(B). $(\Upsilon(h))_0 = h$;
3(B). $\Upsilon(h) \equiv h$ *on* $M \diagdown O$;
4(B). $(\Upsilon(h))_1 = e$ *on* C;
5(B). $\Upsilon(h) \equiv e$ *on* $D \diagdown O'$;
6(B). *if* $h = e|_O$, *then* $(\Upsilon(h))_1 = e|_O$.

In the case of a compact C condition 3 also ensures the continuous dependence of $\Upsilon(h)$ on h in the topologies $\tau = c$ and $\tau = u$, if it is known for $\tau = m$. Therefore Proposition (A), together with Theorem 2, follows from (B) (take D empty and $C = K$).

1.24. Let us derive the Fundamental Theorem, together with the improvement given in 1.17, from Proposition (B). As we explained in 1.17, we may restrict ourselves to the case of a manifold without boundary.

In (B) we put $C = O(C) = M$, and let D be any closed subset of M and O' an arbitrary neighborhood of its boundary. By (B) there exists a majorant f on M such that for each homeomorphism $h \in \Omega_f(e) \cap \Delta(D)$ there is an isotopy $\Upsilon(h)$ with the properties 1–6 (B). From properties 1, 2 and 4 it follows that $\Upsilon(h)$ is a contraction of $\Omega_f(e) \cap \Delta(D)$ into e, and from 5 follows the above-mentioned improvement. Properties 3 and 6 are introduced for use in the proof by induction.

1.25. **Remark 4.** We note, in particular, that if $\mathrm{Fr}\, D \cap \mathrm{Fr}\, C = \Lambda$ then $\Upsilon(h)$ may be so constructed that $(\Upsilon(h))_1$ is the identity on $C \cup D$.

1.26. Proposition (B) is proved in the next three sections. In §2 we reduce it to the local case, which we call the Local Theorem. In §3 a lemma is proved which is the kernel of the whole proof, and which we call the lemma on correction of homeomorphisms. At the beginning of §3 a description is given in general terms of the main ideas of the proof. In §4 the Local Theorem is proved on the basis of the lemma. Finally, §5 is devoted to corollaries and unsolved problems.

§2. Reduction of Proposition (B) to the Local Theorem

2.1. Notation. In R^n we introduce a system of Cartesian coordinates with origin o and axes ox_i, $1 \leq i \leq n$. We shall denote the cube $\{x \mid |x_i| \leq r;\ 1 \leq i \leq n\}$ by I_r^n and the unit cube I_1^n by I^n. We take $n = \dim M$.

2.2. First (subsections 2–5) we shall reduce (B) to the case when C is compact. Represent M as a union $\bigcup_{i=1}^{\infty} K_i$, where the K_i are compact, such that

$$K_i \cap K_{i'} = \Lambda, \quad \text{for} \quad |i-i'|>1, \tag{1}$$

$$\operatorname{Fr} K_i \cap \operatorname{Fr} K_{i'} = \Lambda, \quad \text{for} \quad i \neq i'. \tag{2}$$

From these two conditions it follows that

$$\operatorname{Fr} K_i \subset \operatorname{Int}(K_{i-1} \cup K_{i+1}). \tag{3}$$

Such a representation of M is possible because we have assumed that it is metrizable, and so it is a locally compact paracompact space.

Let $\tilde{O}$ be a neighborhood of C such that

$$[\tilde{O}] \subset \dot{O}, \tag{4}$$

$$\operatorname{Fr} D \cap [\tilde{O}] \subset O' \tag{5}$$

There exists such a neighborhood because C is a closed subset of M in O, and $\operatorname{Fr} D \cap C$ is a closed subset of M in O'. Put $C_i = [\tilde{O}] \cap K_i$. We observe that

$$(\operatorname{Fr} C_i) \cap C \subset \operatorname{Int}(C_{i-1} \cup C_{i+1}). \tag{6}$$

Indeed, $\operatorname{Fr} C_i \cap C \subset \operatorname{Fr} K_i \cap C$, since $C \subset \operatorname{Int}[\tilde{O}]$, and now one applies (3). Choose numbers $\epsilon_i > 0$ so small that

$$[O_{\varepsilon_i}(C_i)] \cap [O_{\varepsilon_{i'}}(C_{i'})] = \Lambda, \quad \text{for} \quad |i-i'|>1, \tag{7}$$

$$[O_{\varepsilon_i}(C_i)] \cap K_{i'} = \Lambda, \quad \text{for} \quad |i-i'|>1, \tag{8}$$

$$O_{\varepsilon_i}(C_i) \subset O, \tag{9}$$

$$\operatorname{Fr} D \cap O_{\varepsilon_i}(C_i) \subset O'. \tag{10}$$

Conditions (7) and (8) can be satisfied by (1), condition (9) by (4), and (10) by (5).

2.3. First consider C_{2i} for $i \geq 1$. Choose numbers $\eta_{2i} > 0$ so small that

$$[O_{\eta_{2i}}(\operatorname{Fr} D \cap C_{2i})] \subset O' \tag{11}$$

and the sets $[O_{\eta_{2i}}(\operatorname{Fr} D \cap C_{2i})]$ are compact and only a finite number intersect each compact set. This is again possible because $\operatorname{Fr} D \cap C_{2i}$ is a compact subset of M lying, by (5), in O'.

Now assume that Proposition (B) is valid in the case of a compact C, and apply it with C_{2i} taken as C, $O_{\epsilon_{2i}}(C_{2i})$ as O, and $O_{\eta_{2i}}(\mathrm{Fr}\, D \cap C_{2i})$ as O'. We obtain a number $\delta_{2i} > 0$, and, for each homeomorphism h which moves the points of $O_{\epsilon_{2i}}(C_{2i})$ less than δ_{2i} and is the identity on D, an isotopy $\Upsilon_{2i}(h)$, satisfying conditions 1–6 of (B) with the given substitutions made. The support of this isotopy is $[O_{\epsilon_{2i}}(C_{2i})]$, and by (7) these sets are mutually disjoint, while by (8) they are compact and only a finite number meet each compact set. Then obviously for every h moving the points of each $O_{\epsilon_{2i}}(C_{2i})$ less than δ_{2i}, and which is the identity on D, an isotopy $\Upsilon_{\mathrm{ev}}(h)$ is defined which on $O_{\epsilon_{2i}}(C_{2i})$ is equal to $\Upsilon_{2i}(h)$, and on $M \setminus \bigcup_{i=1}^{\infty} O_{\epsilon_{2i}}(C_{2i})$ is equal to h. This isotopy has the following properties:

1 (ev). $\Upsilon_{\mathrm{ev}}(h)$ depends continuously on h;

2 (ev). $(\Upsilon_{\mathrm{ev}}(h))_0 = h$;

3 (ev). $\Upsilon_{\mathrm{ev}}(h) \equiv h$ on $M \setminus \bigcup_{i=1}^{\infty} O_{\varepsilon_{2i}}(C_{2i})$;

4 (ev). $(\Upsilon_{\mathrm{ev}}(h))_1 = e$ on $\bigcup_{i=1}^{\infty} C_{2i}$;

5 (ev). $\Upsilon_{\mathrm{ev}}(h) \equiv e$ on $D \setminus \bigcup_{i=1}^{\infty} O_{\eta_{2i}}(\mathrm{Fr}\, D \cap C_{2i})$;

6 (ev). $(\Upsilon_{\mathrm{ev}}(h))_1 = e$ on $O_{\varepsilon_{2i}}(C_{2i})$, if $h = e$ on $O_{\varepsilon_{2i}}(C_{2i})$.

Conditions 2–6 (ev) follow directly from the definition of $\Upsilon_{\mathrm{ev}}(h)$ and the corresponding properties of the isotopies $\Upsilon_{2i}(h)$. Condition 1 (ev) is easily deduced from the fact that each $\Upsilon_{2i}(h)$ depends continuously on h, and that only a finite number of the supports of these isotopies intersect each compact subset of M.

2.4. We now turn to the C_{2i+1} $(i \geq 0)$. Put $D_{\mathrm{od}} = D \setminus \bigcup_{i=1}^{\infty} O_{\eta_{2i}}(\mathrm{Fr}\, D \cap C_{2i})$. We note that

$$\mathrm{Fr}\, D_{\mathrm{od}} \cap \bigcup_{i=1}^{\infty} C_i \subset O'. \tag{12}$$

For, by the condition on the set $[O_{\eta_{2i}}(\mathrm{Fr}\, D \cap C_{2i})]$, their union is closed in M, and lies by (11) in O', and by (5) $\mathrm{Fr}\, D \cap C_i \subset O'$. At the same time $\mathrm{Fr}\, D_{\mathrm{od}} \subset \mathrm{Fr}\, D \cup \bigcup_{i=1}^{\infty} [O_{\eta_{2i}}]$.

Let $D_{2i+1} = C_{2i} \cup C_{2i+2} \cup D_{\mathrm{od}}$, and let the numbers $\eta_{2i+1} > 0$ be so small that

$$O_{\eta_{2i+1}}(\mathrm{Fr}\, D_{\mathrm{od}} \cap C_{2n+1}) \subset O', \tag{13}$$

$$O_{\eta_{2i+1}}(\mathrm{Fr}\,((C_{2i} \cup C_{2i+2}) \setminus D_{\mathrm{od}})) \cap D_{\mathrm{od}} \subset O', \tag{14}$$

$$O_{\eta_{2i+1}}((\mathrm{Fr}\, C_{2i} \cup \mathrm{Fr}\, C_{2i+2}) \cap C_{2i+1}) \cap C \subset C_{2i+1}, \tag{15}$$

$$O_{\eta_{2i+1}}((\mathrm{Fr}\, D_{\mathrm{od}} \setminus (C_{2i} \cup C_{2i+2})) \cap C_{2i+1}) \cap C_{2i} \cap C \cap O_{\varepsilon_{2i+1}}(C_{2i+1}) \subset C_{2i+1}. \tag{16}$$

Condition (13) can be satisfied because $\mathrm{Fr}\, D_{\mathrm{od}} \cap C_{2i+1}$ is a compact subset of M lying, by (12), in O'; condition (14), because $\mathrm{Fr}\,((C_{2i} \cup C_{2i+2}) \setminus D_{\mathrm{od}}) \cap D_{\mathrm{od}} \subset \mathrm{Fr}\, D_{\mathrm{od}} \cap (C_{2i} \cup C_{2i+2})$ and by (12); condition (15), by (6). For (16) we observe that $[\mathrm{Fr}\, D_{\mathrm{od}} \setminus C_{2i}] \cap C_{2i} \subset \mathrm{Fr}\, C_{2i}$, and it remains only to apply (6), since $O_{\epsilon_{2i+1}}(C_{2i+1}) \cap C_{2i-1} = \Lambda$ by (8).

We again apply (B), this time taking C_{2i+1} for C, D_{2i+1} for D, $O_{\epsilon_{2i+1}}(C_{2i+1})$ for O, and $O_{\eta_{2i+1}}(\mathrm{Fr}\, D_{2i+1} \cap C_{2i+1})$ for O'. We find a number $\overline{\delta}_{2i+1}$ such that for each homeomorphism $\overline{h}$ moving the points of $O_{\epsilon_{2i+1}}(C_{2i+1})$ by less than $\overline{\delta}$, and which is the identity on D_{2i+1}, there is an isotopy $\Upsilon_{2i+1}(\overline{h})$ with the properties 1–6(B) when the substitutions have been made. By 1–6 (ev) there exists a number δ_{2i+1} such that if the homeomorphism h moves the points of $O_{\epsilon_{2i}}(C_{2i})$ and $O_{\epsilon_{2i+2}}(C_{2i+2})$ less than δ_{2i+1} then $(\Upsilon_{\mathrm{ev}}(h))_1$ moves the points of $O_{\epsilon_{2i}}(C_{2i}) \cup O_{\epsilon_{2i+1}}(C_{2i+1})$ less than $\overline{\delta}_{2i+1}$. If, moreover, h moves the points of $O_{\epsilon_{2i+1}}(C_{2i+1})$ less than $\overline{\delta}_{2i+1}$, then the same is true for $(\Upsilon_{\mathrm{ev}}(h))_1$, since $(\Upsilon_{\mathrm{ev}}(h))_1 = h$ on $O_{\epsilon_{2i+1}}(C_{2i+1}) \setminus (O_{\epsilon_{2i}}(C_{2i}) \cup O_{\epsilon_{2i+2}}(C_{2i+2}))$. Thus for $(\Upsilon_{\mathrm{ev}}(h))_1$ the isotopy $\Upsilon_{2i+1}((\Upsilon_{\mathrm{ev}}(h))_1)$ is defined. Let h be a homeomorphism which is the identity on D and moves the points of each $O_{\epsilon_{2i}}(C_{2i})$ less than $\overline{\delta}_{2i} = \min(\delta_{2i}, \delta_{2i+1}, \delta_{2i-1})$, and the points of each $O_{\epsilon_{2i+1}}(C_{2i+1})$ less than $\overline{\delta}_{2i+1}$. Then for h the isotopy $\Upsilon_{\mathrm{ev}}(h)$ is defined, and for $(\Upsilon_{\mathrm{ev}}(h))_1$ all the isotopies $\Upsilon_{2i+1}((\Upsilon_{\mathrm{ev}}(h))_1)$, $i \geq 0$, are defined. So these isotopies are defined if $h \in \Omega_f(e) \cap \Delta(D)$, where f is a majorant which is less than $\overline{\delta}_i$ on $O_{\epsilon_{2i}}(C_{2i})$. Since, as we know, the closures of the $O_{\epsilon_{2i}}(C_{2i})$ are compact and only a finite number of them meet each compact set, such a majorant can be constructed. Now, as above for the even case, we can construct for each $h \in \Omega_f(e) \cap \Delta(D)$ an isotopy $\Upsilon_{\mathrm{od}}(h)$ which coincides with $\Upsilon_{2i+1}((\Upsilon_{\mathrm{ev}}(h))_1)$, on $O_{\epsilon_{2i+1}}(C_{2i+1})$, and coincides identically with $(\Upsilon_{\mathrm{ev}}(h))_1$ outside $\bigcup_{i=1}^{\infty} O_{\epsilon_{2i+1}}(C_{2i+1})$. The isotopy $\Upsilon_{\mathrm{od}}(h)$ has the properties:

1 (od). $\Upsilon_{\mathrm{od}}(h)$ depends continuously on h;

2 (od). $(\Upsilon_{\mathrm{od}}(h))_0 = (\Upsilon_{\mathrm{ev}}(h))_1$;

3 (od). $\Upsilon_{\mathrm{od}}(h) \equiv (\Upsilon_{\mathrm{ev}}(h))_1$ on $M \setminus \bigcup\limits_{i=0}^{\infty} O_{\epsilon_{2i+1}}(C_{2i+1})$;

4 (od). $(\Upsilon_{\mathrm{od}}(h))_1 = e$ on $\bigcup\limits_{i=0}^{\infty} C_{2i+1}$;

5 (od). $\Upsilon_{\mathrm{od}}(h) \equiv e$ on $\bigcap\limits_{i=0}^{\infty} (D_{2i+1} \setminus O_{\eta_{2i+1}}(\mathrm{Fr}\, D_{2i+1} \cap C_{2i+1}))$;

6 (od). $(\Upsilon_{\mathrm{od}}(h))_1 = e|_O$, if $h = e|_O$.

Properties 2–5 (od) follow from the definition of $\Upsilon_{\mathrm{od}}(h)$ and the corresponding properties of the $\Upsilon_{2i+1}(h)$; property 1 (od) follows from the continuous dependence of all the $\Upsilon_{2i+1}((\Upsilon_{\mathrm{ev}}(h))_1)$ on h, since only a finite number of the supports of these isotopies intersect each compact set; and 6(od) follows from the fact that, by 6(ev), if $h = e$ then $(\Upsilon_{\mathrm{ev}}(h))_1 = e$, and so all the homeomorphisms $\Upsilon_{2i+1}((\Upsilon_{\mathrm{ev}}(h))_1) = e$ on O.

2.5. In view of 2(od), the composition $\Upsilon_{\mathrm{ev}}(h) \circ \Upsilon_{\mathrm{od}}(h)$ is defined, and this we take as $\Upsilon(h)$. Let us show that it has the required properties. Property 1(B) follows from 1(ev) and 1(od); property 2(B) is 2(ev) and 6(B) is 6(od); 3(B) follows from 3(od), 3(ev), and (9). We now verify 4(B). First, by 4(od), $(\Upsilon(h))_1 = (\Upsilon_{\mathrm{od}}(h))_1 = e$ on $\bigcup_{i=0}^{\infty} C_{2i+1}$ and so on $\bigcup_{i=0}^{\infty} (C \cap K_{2i+1})$. On the other hand $(\Upsilon(h))_1 = e$ on $C \setminus \bigcup_{i=0}^{\infty} O_{\epsilon_{2i+1}}(C_{2i+1})$, by 4(ev) and 3(od).

Noting that $O_{\epsilon_{2i+1}}(C_{2i+1}) \cap O_{\epsilon_{2i'+1}}(C_{2i'+1}) = \Lambda$ if $i \neq i'$ (see (7)), we see that it is sufficient to consider $(\Upsilon(h))_1$ on

$$C \cap O_{\varepsilon_{2i+1}}(C_{2i+1}) \setminus C_{2i+1}$$
$$\subset ((C \cap C_{2i} \cap O_{\varepsilon_{2i+1}}(C_{2i+1})) \setminus C_{2i+1}) \cup ((C \cap C_{2i+2} \cap O_{\varepsilon_{2i+1}}(C_{2i+1})) \setminus C_{2i+1}).$$

We consider only the first term, since the second is similar. By 5(od), $(\Upsilon(h))_1 = (\Upsilon_{\mathrm{od}}(h))_1 = (\Upsilon_{2i+1}(h))_1 = e$ on

$$O_{\varepsilon_{2i+1}}(C_{2i+1}) \cap C \cap (D_{2i+1} \setminus O_{\eta_{2i+1}}(\mathrm{Fr}\, D_{2i+1} \cap C_{2i+1})) \supset O_{\varepsilon_{2i+1}}(C_{2i+1}) \cap C^{*}$$
$$\cap (C_{2i} \setminus O_{\eta_{2i+1}}(\mathrm{Fr}\, D_{2i+1} \cap C_{2i+1}) \supset O_{\varepsilon_{2i+1}}(C_{2i+1}) \cap C$$
$$\cap (C_{2i} \setminus O_{\eta_{2i+1}}((\mathrm{Fr}\, C_{2i} \cup \mathrm{Fr} C_{2i+2}) \cap C_{2i+1}) \setminus O_{\eta_{2i+1}}((\mathrm{Fr}\, D_{\mathrm{od}} \setminus (C_{2i} \cup C_{2i+2})) \cap C_{2i+1})).$$

But by (16) the last term lies in C_{2i+1}, and since $(\Upsilon(h))_1 = e$ on C_{2i+1} we obtain that $(\Upsilon(h))_1 = e$ on $O_{\epsilon_{2i+1}}(C_{2i+1}) \cap C \cap (C_{2i}) \setminus O_{\eta_{2i+1}}((\mathrm{Fr}\, C_{2i} \cup \mathrm{Fr}\, C_{2i+2}) \cap C_{2i+1}))$, and, by (15), on $O_{\epsilon_{2i+1}}(C_{2i+1}) \cap C \cap (C_{2i} \setminus C_{2i+1})$, as required.

We now verify 5(B). By 5(ev), for $t \leq \frac{1}{2}$ we have $(\Upsilon(h))_t = (\Upsilon_{\mathrm{ev}}(h))_{2t} = e$ on D_{od}, and so, by (11), on $D \setminus O'$. For $\frac{1}{2} \leq t \leq 1$, by 5(od), $(\Upsilon(h))_t = (\Upsilon_{\mathrm{od}}(h))_{2t-1} = e$ on $D_{\mathrm{od}} \setminus \bigcup_{i=0}^{\infty} O_{\eta_{2i+1}}(\mathrm{Fr}\, D_{2i+1} \cap C_{2i+1})$, and since $D_{\mathrm{od}} \supset D \setminus O'$ it is sufficient to show that for all i

$$D_{\mathrm{od}} \cap O_{\eta_{2i+1}}(\mathrm{Fr}\, D_{2i+1} \cap C_{2i+1}) \subset O'.$$

But

$$D_{\mathrm{od}} \cap O_{\eta_{2i+1}}(\mathrm{Fr}\, D_{2i+1} \cap C_{2i+1})$$
$$\subset O_{\eta_{2i+1}}(\mathrm{Fr}\, D_{\mathrm{od}} \cap C_{2i+1}) \cup (D_{\mathrm{od}} \cap O_{\eta_{2i+1}}(\mathrm{Fr}\,(C_{2i} \cup C_{2i+2} \setminus D_{\mathrm{od}}))),$$

and (13) and (14) can be applied.

Thus we may suppose in the statement of (B) that C is compact.

2.6. Now we shall reduce (B) to the case when $M = R^n$. Let $\{Q_i, q_i\}_{i=1}^{k}$ be a finite covering of the component set C by Euclidean neighborhoods; that is, q_i: $R^n \to M$ is a homeomorphic mapping and $Q_i = q_i R^n$, with $C \subset \bigcup_{i=1}^{k} Q_i$. We suppose that the $[Q_i]$ are compact and that

$$\bigcup_{i=1}^{k} Q_i \subset O. \tag{17}$$

We shall construct the required isotopy $\Upsilon(h)$ as the composition of isotopies $\Psi_i(h)$, each having support in one of the Q_i and satisfying specific conditions similar to those of (B). By means of q_i the construction of $\Psi_i(h)$ will be transferred to R^n. This reduction will be carried out in 2.7–2.10.

2.7. In constructing the isotopies $\Psi_i(h)$ it will be convenient to assume that

$$\bigcup q_i I^n \supset C, \tag{18}$$

which of course we can do without loss of generality. We shall also assume that the Q_i lie in a sufficiently small neighborhood of the compact set C so that

$$\mathrm{Fr}\, D \cap \bigcup_{i=1}^{k} [Q_i] \subset O'. \tag{19}$$

Condition (19) can be satisfied because $\operatorname{Fr} D \cap C$ is a compact subset of O'.

Take k more neighborhoods $O_i = O_i(\operatorname{Fr} D \cap C)$ such that

$$[O_i] \subset O_{i+1} \subset O', \tag{20}$$

which is possible because $\operatorname{Fr} D \cap C$ is a compact subset of O'. Further, let

$$\operatorname{Fr} D \cap \bigcup_{i=1}^{k} [Q_i] \subset O', \tag{21}$$

which is permissible by (19).

Also take numbers $\gamma_i \geq 0$, $1 \leq i \leq k$, such that

$$0 < \gamma_i < \gamma_{i-1} \leqslant \gamma_1 \leqslant 1/4 \quad \text{for } i \geqslant 2. \tag{22}$$

For $1 \leq i \leq k$ we shall successively construct numbers $\delta_i > 0$, $\delta_{i+1} < \delta_i$, and, for each homeomorphism h that moves the points of O less than δ_i and is the identity D, an isotopy $\Psi_i(h)$, in such a way that the following conditions are satisfied:

1 (Ψ). $\Psi_i(h)$ depends continuously on h;

2 (Ψ). $(\Psi_i(h))_0 = (\Psi_{i-1}(h))_1$ for $i \geq 2$, $(\Psi_1(h))_0 = h$;

3 (Ψ). $\Psi_i(h) \equiv (\Psi_{i-1}(h))_1$ for $i \geq 2$ on $M \setminus Q_i$, $\Psi_1(h) \equiv h$ on $M \setminus Q_1$;

4 (Ψ). $(\Psi_i(h))_1 = e$ on $\bigcup_{i'=1}^{i} q_{i'} I^n_{1+\gamma_i}$;

5 (Ψ). $\Psi_i(h) \equiv e$ on $D \setminus O_i$;

6 (Ψ). $(\Psi_i(h))_1 = e$ on O if $h = e$ on O.

2.8. Let us show that if numbers δ_i and isotopies Ψ_i with these properties have been constructed then we may take the required function f as equal to δ_k on the whole neighborhood O, and the required isotopy $\Upsilon(h)$, for homeomorphisms h which move the points of O less than δ_k and are the identity on D, as the composition $\Psi_1(h) \circ \cdots \circ \Psi_k(h)$.

We note that for such a homeomorphism all the isotopies $\Psi_i(h)$ are defined, and by $2(\Psi)$ their composition is defined, with $(\Upsilon(h))_0 = h$, so that 2(B) is satisfied. Properties 1(B) and 6(B) follow immediately from $1(\Psi)$ and $6(\Psi)$ respectively, property 3(B) follows from $3(\Psi)$ by (17), 4(B) from $4(\Psi)$ by (18), since $(\Upsilon(h))_1 = (\Psi_k(h))_1$, and 5(B) from $5(\Psi)$ by (20).

2.9. *Arguing by induction, suppose that* δ_{i-1} *and* Ψ_{i-1} *have already been constructed.* Thus for each homeomorphism h that moves the points of O less than δ_{i-1} and is the identity on D we have a homeomorphism h_{i-1} (equal to h for $i = 1$ and to $(\Psi_{i-1}(h))_1$ for $i \geq 2$) with the properties

1(h). h_{i-1} depends continuously on h;

2(h). $h_{i-1} = e$ on $\bigcup_{i'=1}^{i-1} q_{i'} I^n_{1+\gamma_i} \cup (D \setminus O_{i-1})$;

3(h). $h_{i-1} = e$ on O if $h = e$ on O.

If $i \geq 2$ then 1(h) follows from $1(\Psi)$, 2(h) from $4(\Psi)$, and 3(h) from $6(\Psi)$. If $i = 1$ then 1(h) and 3(h) are trivial, while 2(h) reduces to the condition that $h = e$ on D.

Thus our construction can be started.

2.10. We pass to the construction of $\Psi_i(h)$ in R^n by means of the homeomorphism q_i. We shall assume the following proposition, which will be proved later (see 2.11).

Proposition (C). *If $\widetilde{D}$ is a closed subset of R^n, then for every neighborhood $\widetilde{O} = \widetilde{O}(\operatorname{Fr} \widetilde{D} \cap I^n_2)$*

there exists a $\widetilde{\delta}$ *such that for each homeomorphic* $\widetilde{\delta}$*-shift* $\widetilde{g}\colon I_2^n \to R^n$ *which is the identity on* $\widetilde{D} \cap I_2^n$ *there is an isotopy* $\widetilde{\Psi}(\widetilde{g})$ *of the space* R^n *such that*

1 (C). $\widetilde{\Psi}(\widetilde{g})$ *depends continuously on* $\widetilde{g}$;

2 (C). $(\widetilde{\Psi}(\widetilde{g}))_0 = e$;

3 (C). $\widetilde{\Psi}(\widetilde{g}) \equiv e$ *on* $R^n \setminus \widetilde{g} I_2^n$;

4 (C). $(\widetilde{\Psi}(\widetilde{g}))_1 = \widetilde{g}^{-1}$ *on* $\widetilde{g} I^n_{1\frac14}$;

5 (C). $\widetilde{\Psi}(\widetilde{g}) \equiv e$ *on* $\widetilde{D} \setminus \widetilde{O}$;

6 (C). $\widetilde{\Psi}(\widetilde{g}) \equiv e$ *if* $\widetilde{g} = e$ *on* I_2^n.

We take as $\widetilde{D}$ the set $g_i^{-1}((\mathbf{U}_{i'=1}^{i-1} g_{i'} I^n_{1+\gamma_{i-1}}) \cup (D \setminus O_{i-1}))$, and we let $\widetilde{O}$ be a neighborhood of $\operatorname{Fr} \widetilde{D} \cap I_2^n$ so small that

$$q_i(\widetilde{D} \setminus \widetilde{O}) \supset D \setminus O_i, \tag{23}$$

$$q_i(\widetilde{D} \setminus \widetilde{O}) \supset \mathbf{U}_{i'=1}^{i-1} q_{i'} I^n_{1+\gamma_i} \cap [Q_i]. \tag{24}$$

Condition (23) can be satisfied because $q_i(\operatorname{Fr} \widetilde{D} \cap I_2^n)$ is a compact subset of M not intersecting $D \setminus O_i$. In fact,

$$q_i(\operatorname{Fr} \widetilde{D} \cap I_2^n) \subset \left(\operatorname{Fr}\left(\left(\mathbf{U}_{i'=1}^{i-1} q_{i'} I^n_{1+\gamma_i}\right) \cup \left(D \setminus O_{i-1}\right)\right)\right) \cap [Q_i]$$

$$\subset (\operatorname{Fr}(D \setminus O_{i-1}) \cap [Q_i]) \cup ([\operatorname{Fr}\left(\mathbf{U}_{i'=1}^{i-1} q_{i'} I^n_{1+\gamma_i}\right) \setminus (D \setminus O_{i-1})] \cap [Q_i]).$$

But $[O_{i-1}] \subset O_i$ (see (20)), and therefore if the intersection of the second term with $D \setminus O_i$ is nonempty then it lies in the first. At the same time $\operatorname{Fr}(D \setminus O_{i-1}) \cap [O_i] \subset (\operatorname{Fr} D \cap [Q_i]) \cup \operatorname{Fr} O_{i-1}$. Here both terms lie in O_i (the first by (20) and (21), and the second by (20)), and so neither intersects $D \setminus O_i$. Condition (24) can be satisfied because $\mathbf{U}_{i'=1}^{i-1} q_{i'} I^n_{1+\gamma_i} \subset \operatorname{Int} \mathbf{U}_{i'=1}^{i-1} q_{i'} I^n_{1+\gamma_{i-1}}$, by (22). We now find a number $\widetilde{\delta}$ by (C). Also let $\widetilde{\delta} < 1$. Now if $\widetilde{g}$ is a $\widetilde{\delta}$-shift of I_2^n in R^n then $g I_2^n \subset I_3^n$. Let $\overline{\delta} > 0$ be so small that if the diameter of a set in $q_i I_3^n$ is less than $\overline{\delta}$ then that of its inverse image in I_3^n is less than $\widetilde{\delta}$. Further, let $\overline{\delta}$ be so small that $O_{\overline{\delta}}(q_i I_2^n) \subset q_i I_3^n$. Then if a homeomorphism $\widetilde{h}$ of the manifold moves the points of Q_i less than $\overline{\delta}$, the homeomorphism $\widetilde{g} = q_i^{-1} \widetilde{h} q_i$ is defined on I_2^n, moves points less than $\widetilde{\delta}$, and by 2(h) and the definition is the identity on $\widetilde{D}$. Thus the isotopy $\widetilde{\Psi}(\widetilde{g})$ is defined, with properties 1–6(C).

By 1(h) and 3(h), there exists a positive number, which we take as δ_i, less than δ_{i-1} and such that if $h \in \Omega_{[Q_i], \delta_i}$ then $h_{i-1} \in \Omega_{q_i I_3^n, \overline{\delta}}(e)$. Put $\widetilde{g} = q_i^{-1} h_{i-1} q_i$ and take for $\Psi_i(h)$ a homeomorphism identically equal to h_{i-1} on $M \setminus h_{i-1} q_i I$ and equal to $q_i \widetilde{\Psi}(\widetilde{g}) q_i^{-1} h_{i-1} = q_i \widetilde{\Psi}(\widetilde{g}) \widetilde{g}\, q_i^{-1}$ on $q_i I_2^n$. It is clear that the two definitions agree on $\operatorname{Fr} q_i I_2^n$, by condition 3(C), and so we obtain an isotopy. Let us verify that $\Psi_i(h)$ satisfies all the conditions 1–6(Ψ). Condition 1(Ψ) follows from 1(h) and 1(C).

Condition $2(\Psi)$ follows from the fact that, according to the definition of $\Psi_i(h)$ and property $2(C)$, we have $(\Psi_i(h))_0 = h_{i-1}$, and it remains only to recall that by definition h_{i-1} is $(\Psi_{i-1}(h))_1$ for $i > 1$. Condition $3(\Psi)$ follows from the fact that, by $3(C)$ and the definition of $\Psi_i(h)$, we have $\Psi_i(h) = h_{i-1}$ outside $q_i I^n_2$, and so outside Q_i. Condition $4(\Psi)$: from $4(C)$ it follows that $(\Psi_i(h))_1 = e$ on $q_i I^n_{1\frac14}$, and so on $q_i I^n_{1+\gamma_i}$, since $\gamma_i < \frac14$ by (22). Further, by $5(C)$ and the definition we have $(\Psi_i(h))_1 = e$ on $q_i(\widetilde{D}\setminus\widetilde{O})$, and so, by (24), on $\bigcup_{i'=1}^{i-1} q_{i'} I^n_{1+\gamma_i}$. Thus $4(\Psi)$ is satisfied. Moreover, by (23), $(\Psi_i(h))_1 = e$ on $D\setminus O_i$, so that $5(\Psi)$ is also satisfied. Condition $6(\Psi)$ follows from 6(C), 3(h), and the definition of $\Psi_i(h)$.

2.11. We pass to the proof of Proposition (C). In the remaining part of this section we reduce this proposition to an assertion (see 2.15) which, as we have said, is the local case of our theorem and will be proved in §4.

We introduce some notation. If T is a triangulation of the space R^n then T'' denotes the second barycentric subdivision. By $\mathrm{St}_T X$ we denote the union of the closed simplexes of T which either are incident on simplexes of X, if X is a subcomplex, or intersect X, if X is a subset of R^n.

2.12. We take a triangulation T of the space R^n so fine that

$$\mathrm{St}_{T''}\mathrm{St}_T(\widetilde{D}\setminus\widetilde{O}) \cap (\mathrm{Fr}\ \widetilde{D}\cap I^n_2) = \Lambda, \tag{25}$$

$$\mathrm{St}_{T''}\mathrm{St}_T I^n_{1^1/_4} \subset \mathrm{Int}\ I^n_{1,5}. \tag{26}$$

For each open simplex $\sigma \in T$ we take the cell $z = z(\sigma) = (\mathrm{St}_{T''}(\sigma)\setminus(\mathrm{St}_{T''}(\partial\sigma))$. We note that $z \cap z' \neq \Lambda$ if and only if the corresponding simplexes are incident. We enumerate the simplexes of $\mathrm{St}_T I^n_{1\frac14}$, first enumerating those belonging to $\mathrm{St}_T(\widetilde{D}\setminus\widetilde{O})$: $\sigma_1, \sigma_2, \cdots, \sigma_d$, and then the remaining ones in order of increasing dimensions: $\sigma_{d+1}, \sigma_{d+2}, \cdots, \sigma_s$. We note that if $z_i \cap (\widetilde{D}\setminus\widetilde{O}) \neq \Lambda$ then $i \leq d$.

Choose numbers $\widetilde{\eta}_i \geq 0$, $d+1 \leq i \leq s$, so small that

$$O_{\widetilde{\eta}_i}(z_i) \subset I^n_{1,5} \text{ (see (26))}; \tag{27}$$

$$O_{\widetilde{\eta}_i}(z_i) \cap O_{\widetilde{\eta}_{i'}}(z_{i'}) = \Lambda, \quad \text{if} \quad z_i \cap z_{i'} = \Lambda; \tag{28}$$

$$\text{if } O_{\widetilde{\eta}_i}(z_i) \cap (\widetilde{D}\setminus\widetilde{O}) \neq \Lambda, \text{then } i \leqslant d. \tag{29}$$

We construct a sequence of numbers $\widetilde{\delta}_i > 0$ such that

$$\widetilde{\delta}_i < \widetilde{\delta}_{i-1} < {}^1/_2, \tag{30}$$

and for each homeomorphic $\widetilde{\delta}_i$-shift $\widetilde{g}\colon I^n_2 \to R^n$ which is the identity on $\widetilde{D}$, we construct an isotopy $\widetilde{\Psi}_i(\widetilde{g})$ such that the first d of these isotopies are the identity and the following conditions are satisfied:

$1(\widetilde{\Psi})$. $\widetilde{\Psi}_i(\widetilde{g})$ depends continuously on $\widetilde{g}$;

$2(\widetilde{\Psi})$. $(\widetilde{\Psi}_i(\widetilde{g}))_0 = (\Psi_{i-1}(\widetilde{g}))_1$, if $i > d+1$, $(\widetilde{\Psi}_i(\widetilde{g}))_0 = e$ if $i = d+1$;

$3(\widetilde{\Psi})$. $\widetilde{\Psi}_i(\widetilde{g}) \equiv (\widetilde{\Psi}_{i-1}(\widetilde{g}))_1$ for $i > d+1$ on $\widetilde{g}(R^n\setminus O_{\widetilde{\eta}_i}(z_i))$,

$\tilde{\Psi}_i(\tilde{g}) \equiv e$ for $i = d+1$ on $\tilde{g}(R^n \setminus O_{\tilde{\eta}_{d+1}}(z_{d+1}))$;

4 $(\tilde{\Psi})$. $(\tilde{\Psi}_i(\tilde{g}))_1 = \tilde{g}^{-1}$ on $\tilde{g} z_i$;

5 $(\tilde{\Psi})$. $\tilde{\Psi}_i(\tilde{g}) = (\tilde{\Psi}_{i-1}(\tilde{g}))_1$ on $\tilde{g}(\mathrm{St}_{T''}\partial\sigma_i)$;

6 $(\tilde{\Psi})$. $(\tilde{\Psi}_i(\tilde{g}))_1 = e$, if $\tilde{g} = e$ on I^n_2.

2.13. Put $\tilde{\delta} \le \min(\tfrac{1}{2}, \tilde{\delta}_s)$. Then for each homeomorphic $\tilde{\delta}$-shift $\tilde{g}$: $I^n_2 \to R^n$, all the isotopies $\tilde{\Psi}_i(\tilde{g})$ are defined, and by $2(\tilde{\Psi})$ their composition is defined, and we take it for $\tilde{\Psi}(\tilde{g})$. Moreover $(\tilde{\Psi}(\tilde{g}))_0 = e$, so that 2(C) is satisfied. Conditions 1(C) and 6(C) obviously follow from $1(\tilde{\Psi})$ and $6(\tilde{\Psi})$. Condition 3(C) follows from the fact that, according to $3(\tilde{\Psi})$, the $O_{\tilde{\eta}_i}(z_i)$ are supports of the isotopies $\tilde{\Psi}_i(\tilde{g})$, and, by (27), $O_{\tilde{\eta}_i}(z_i) \subset I^n_{1.5}$, and so $\tilde{\Psi}_i(\tilde{g}) \equiv e$ on $R^n \setminus I^n_{1.5}$. But $\tilde{g} I^n_2 \supset I^n_{1.5}$ since $\tilde{\delta} < \tfrac{1}{2}$, and hence $\tilde{\Psi}_i(\tilde{g}) = e$ on $R^n \setminus \tilde{g} I^n_2$.

Let us verify 4(C). First, by $4(\Psi)$, $(\tilde{\Psi}_i(\tilde{g}))_1 = \tilde{g}^{-1}$ on $\tilde{g} z_i$. If $i' > i$, either $O_{\tilde{\eta}_{i'}}(z_i)$ does not intersect z_i or σ_i lies in $\partial\sigma_{i'}$. In both cases $\tilde{\Psi}_{i'}(\tilde{g}) \equiv (\tilde{\Psi}_{i'-1}(\tilde{g}))_1$ on z_i (by $3(\tilde{\Psi})$ in the first case, and by $5(\tilde{\Psi})$ in the second). By induction, $\tilde{\Psi}_{i'}(\tilde{g}) \equiv \tilde{g}^{-1}$ on $\tilde{g} z_i$, and so $(\tilde{\Psi}(\tilde{g}))_1 = (\tilde{\Psi}_s(\tilde{g}))_1 = \tilde{g}^{-1}$ on all the z_i. But evidently $\bigcup_{i=1}^s z_i \supset I^n_{1\frac{1}{4}}$.

It remains to verify 5(C). As we have seen, $\tilde{\Psi}(\tilde{g}) \equiv e$ on $R^n \setminus \bigcup_{i=1}^s O_{\tilde{\eta}_i}(z_i)$. If $O_{\tilde{\eta}_i}(z_i)$ intersects $\tilde{D} \setminus \tilde{O}$ then $i \le d$ by (29), and all the initial isotopies $\tilde{\Psi}_i(\tilde{g})$ for $1 \le i \le d$ are the identity.

2.14. We construct the $\tilde{\delta}_i$ and $\tilde{\Psi}_i(\tilde{g})$ successively. Arguing by induction, we assume that $\tilde{\delta}_{i-1}$ and $\tilde{\Psi}_{i-1}(\tilde{g})$ have already been constructed.

Thus for each homeomorphic $\tilde{\delta}_{i-1}$-shift $\tilde{g}$: $I^n_2 \to R^n$ we have a homeomorphism $\tilde{h}_{i-1}$: $R^n \to R^n$ equal to e for $i = 1$ and equal to $(\tilde{\Psi}_{i-1}(\tilde{g}))_1$ for $i > 1$, with the properties

1 $(\tilde{h})$. $\tilde{h}_{i-1}$ depends continuously on $\tilde{g}$;

2 $(\tilde{h})$. $\tilde{h}_{i-1} = \tilde{g}^{-1}$ on $\tilde{g}\mathrm{St}_{T''}\partial\sigma_i$;

3 $(\tilde{h})$. $\tilde{h}_{i-1} = e$, if $\tilde{g} = e$ on I^n_2.

If $i = 1$ then these properties are obvious. If $i \le d+1$ they are also obvious, since each of the first d isotopies is the identity. Let $i > d+1$. Properties $1(\tilde{h})$ and $3(\tilde{h})$ follow immediately from $1(\tilde{\Psi})$ and $6(\tilde{\Psi})$. Property $2(\tilde{h})$ follows, as we saw above, from $4(\tilde{\Psi})$ and $5(\tilde{\Psi})$. We must now construct $\tilde{\delta}_i$ and $\tilde{\Psi}_i$ for $i > d$. Consider the homeomorphism $\tilde{\tilde{g}}_{i-1}$: $O_{\tilde{\eta}_i}(z_i) \to R^n$ equal to $\tilde{h}_{i-1}\tilde{g}$ on $O_{\tilde{\eta}_i}(z_i)$. It obviously has the properties

1 $(\tilde{\tilde{g}})$. $\tilde{\tilde{g}}_{i-1}$ depends continuously on $\tilde{g}$;

2 $(\tilde{\tilde{g}})$. $\tilde{\tilde{g}}_{i-1} = e$ on $\mathrm{St}_{T''}\partial\sigma_i$;

3 $(\tilde{\tilde{g}})$. $\tilde{\tilde{g}}_{i-1} = e$, if $\tilde{g} = e$ on I^n_2.

We shall construct a number $\tilde{\tilde{\delta}}_i > 0$, and an isotopy $\tilde{\tilde{\Psi}}_i(\tilde{\tilde{g}})$ for each homeomorphic $\tilde{\tilde{\delta}}_i$-shift $\tilde{\tilde{g}}$: $O_{\tilde{\eta}_i}(z_i) \to R^n$ with properties $1\text{–}3(\tilde{\tilde{g}})$, such that

1 $(\tilde{\tilde{\Psi}})$. $\tilde{\tilde{\Psi}}_i(\tilde{\tilde{g}})$ depends continuously on $\tilde{\tilde{g}}$;

2 $(\tilde{\tilde{\Psi}})$. $(\tilde{\tilde{\Psi}}_i(\tilde{\tilde{g}}))_0 = e$;

3 $(\tilde{\tilde{\Psi}})$. $\tilde{\tilde{\Psi}}_i(\tilde{\tilde{g}}) \equiv e$ on $\tilde{\tilde{g}}(R^n \setminus O_{\tilde{\eta}_i}(z_i))$;

4 $(\tilde{\tilde{\Psi}})$. $(\tilde{\tilde{\Psi}}_i(\tilde{\tilde{g}}))_1 = \tilde{\tilde{g}}^{-1}$ on $\tilde{\tilde{g}} z_i$;

5 $(\tilde{\tilde{\Psi}})$. $\tilde{\tilde{\Psi}}_i(\tilde{\tilde{g}}) \equiv e$ on $\mathrm{St}_{T''} \partial\sigma_i$;

6 $(\tilde{\tilde{\Psi}})$. $(\tilde{\tilde{\Psi}}_i(\tilde{\tilde{g}})) = e$, if $\tilde{\tilde{g}} = e$ on $O_{\tilde{\eta}_i}(z_i)$.

By conditions $1(\tilde{\tilde{g}})$ and $3(\tilde{\tilde{g}})$ there exists a number, which we take as $\tilde{\tilde{\delta}}_i$, less than $\tilde{\delta}_{i-1}$ and such that if $\tilde{g}$ is a homeomorphic $\tilde{\delta}_i$-shift of I^n_2 in R^n then the mapping $\tilde{\tilde{g}}_{i-1} = \tilde{h}_{i-1}\tilde{g}$ (which is obviously defined, since $\tilde{\delta}_i < \tilde{\delta}_{i-1}$) is a homeomorphic $\tilde{\tilde{\delta}}_i$-shift on $O_{\tilde{\eta}_i}(z_i)$. Then the isotopy $\tilde{\tilde{\Psi}}_i(\tilde{\tilde{g}}_{i-1})$ is defined, and we take as the required isotopy $\tilde{\Psi}_i(\tilde{g})$ the isotopy identically equal to $\tilde{h}_{i-1} = (\tilde{\Psi}_{i-1}(\tilde{g}))_1$ outside $\tilde{g} O_{\tilde{\eta}_i}(z_i)$ and equal to $\tilde{\tilde{\Psi}}_{i-1}(\tilde{\tilde{g}}_{i-1})\tilde{h}_{i-1}$ on $\tilde{g} O_{\tilde{\eta}_i}(z_i)$.

Properties 1–6$(\tilde{\Psi})$ follow immediately from the definition, properties 1–6$(\tilde{\tilde{\Psi}})$, and the properties of the homeomorphism $\tilde{h}_{i-1}$ and $\tilde{\tilde{g}}_{i-1}$.

2.15. Now we carry out the last step in the reduction of proposition (B) to the Local Theorem.

Let $p = \dim \sigma_i$. Represent R^n as $R^p \times \bar{R}^{n-p}$, where R^p is the coordinate hyperplane spanned by the first p axes and $\bar{R}^{n-p}$ is that spanned by the last $n-p$. Let I^p_r and $\bar{I}^{n-p}_r$ be the cubes defined for R^p and $\bar{R}^{n-p}$ just as I^n_r is for R^n.

In an obvious way a homeomorphic mapping $q\colon R^n \to O_{\tilde{\eta}_i}(z_i)$ can be constructed such that

1 (q). $qI^n = z_i$;

2 (q). $q(R^n \setminus (I^p \times \bar{R}^{n-p})) \subset \mathrm{St}_{T''} \partial\sigma_i$;

3 (q). $q(I^n \times \bar{R}^{n-p}) \subset O_{\tilde{\eta}_i}(z_i) \setminus \mathrm{St}_{T''} \partial\sigma_i$.

With the help of the homeomorphism q the construction of $\tilde{\tilde{\Psi}}_i(\tilde{\tilde{g}})$ reduces to the following proposition.

2.16. **Local Theorem.** *There is a number $\delta > 0$ such that for each homeomorphic δ-shift g: $I^n_2 \to R^n$ which is the identity on $I^n_2 \setminus (I^p \times \bar{I}^{n-p}_2)$ there exists an isotopy* $\mathrm{H}(q)$ *such that*

1 (L). $\mathrm{H}(g)$ *depends continuously on* g;

2 (L). $(\mathrm{H}(g))_0 = e$;

3 (L). $\mathrm{H}(g) \equiv e$ *on* $R^n \setminus g(I^p \times \bar{I}^{n-p}_2)$;

4 (L). $(\mathrm{H}(g))_1 = g^{-1}$ *on* gI^n_1;

5 (L). $(\mathrm{H}(g))_1 = e$, *if* $g = e$ *on* I^n_2.

2.17. Let us show how to construct $\tilde{\tilde{\delta}}_i$ and $\tilde{\tilde{\Psi}}_i(\tilde{\tilde{g}})$, using this result. Since q is uniformly continuous on I^n_2, there exists a number, which we take as $\tilde{\tilde{\delta}}_i$, such that if the diameter of a set in qI^n_2 is less than $\tilde{\tilde{\delta}}_i$ then that of its preimage in I^n_2 is less than δ. Suppose also that $\tilde{\tilde{\delta}}_i < \rho(\mathrm{Fr}\, O_{\tilde{\eta}_i}(z_i), qI^n_2)$. Then for a $\tilde{\tilde{\delta}}_i$-shift $\tilde{\tilde{g}}_i\colon I^n_2 \to R^n$ we have

$$\tilde{\tilde{g}}_i(qI^n_2) \subset O_{\tilde{\eta}_i}(z_i). \tag{31}$$

Put $g = q^{-1}\tilde{\tilde{g}}q$. By (31) this homeomorphism is defined throughout I^n_2, and by the choice of $\tilde{\tilde{\delta}}_i$ it is a δ-shift of I^n_2 in R^n. By the Local Theorem we find $\mathrm{H}(g)$, and we take as $\tilde{\tilde{\Psi}}(\tilde{\tilde{g}})$ the isotopy equal to

E outside $\tilde{\tilde{g}} q I_2^n = q g I_2^n$ and to $q\mathrm{H}(g)q^{-1}$ on $\tilde{\tilde{g}} q I_2^n$. Then $2(\tilde{\tilde{\Psi}})$ and $6(\tilde{\tilde{\Psi}})$ follow from 1(L), 2(L) and 5(L); $3(\tilde{\tilde{\Psi}})$ from 3(L), since $\tilde{\tilde{g}}(qI_2^n) \subset O_{\tilde{\eta}_i}$; and $4(\tilde{\tilde{\Psi}})$ from 4(L) and the fact that, by 1(q), $qI^n = z_i$. Finally, $5(\tilde{\tilde{\Psi}})$ follows from 3(L), since $q^{-1}(\mathrm{St}_{T''}\partial\sigma_i) \cap O_{\tilde{\eta}_i} = R^n \setminus I^p \times \bar{I}_2^{n-p}$, and by 3(L), $\mathrm{H} \equiv e$ on $R^n \setminus g(I^p \times \bar{R}^{n-p})$, but

$$g(I^p \times \bar{R}^{n-p}) = q^{-1}\tilde{\tilde{g}} q(I^p \times \bar{R}_2^{n-p}) \subset q^{-1}\tilde{\tilde{g}}(qI_2^n \setminus \mathrm{St}_{T''}\partial\sigma_i)$$
$$\subset q^{-1}(qI_2^n \setminus \mathrm{St}_{T''}\partial\sigma_i) \subset I^p \times \bar{R}^{n-p},$$

by 3(q), $2(\tilde{\tilde{g}})$, and 2(q).

§3. The idea of the proof of the Local Theorem and the lemma on correction of homeomorphisms

3.1. We first introduce some notation which will be used right up to the end of the proof. We define

$$R_{i,d} = \{x \mid x_i = d\}; \quad R_{i,d}^+ = \{x \mid x_i \geqslant d\}; \quad R_{i,d}^- = \{x \mid x_i \leqslant d\};$$

$$R_i(d_1, d_2) = R_{i,d_1}^+ \cap R_{i,d_2}^-; \quad R_{i,d}^\varepsilon = R_i(d-\varepsilon, d+\varepsilon);$$
$$I_i(d; r) = R_{i,d} \cap I_r^n; \quad I_i^\varepsilon(d; r) = R_{i,d}^\varepsilon \cap I_r^n; \quad \Pi_i(d_1, d_2; r) = R_i(d_1, d_2) \cap I_r^n.$$

3.2. We now make some preliminary remarks which will, we hope, help the reader to understand the main idea and the plan of the proof.

First suppose that in R^n we are given three $(n-1)$-dimensional hyperplanes orthogonal to some coordinate axis: R_{i,a_1}, R_{i,a_2}, R_{i,a_3}, $a_1 < a_2 < a_3$, and let a homeomorphism $h\colon R^n \to R^n$ be given, such that for some given number $\epsilon > 0$ we have $hR_{i,a_j} \subset R_{i,a_j}^\epsilon$ for all values of j.

Let $c = (a_1 + a_2)/2$, where $2\epsilon < c - a_1$. Consider the problem of constructing an isotopy ω of R^n satisfying the following conditions:

1) $\omega_0 = e$;

2) $\omega \equiv e$ on $R^n \setminus R_i(a_1, a_2)$;

3) $\omega_1 = e$ on $R^n \setminus R_i(a_1, a_2)$;

4) $\omega_1 h R_{i,c} \subset R_{i,c}^\varepsilon$,

the latter condition being satisfied in some given finite part of the space. We shall henceforth omit stipulations of the type "in a sufficiently large finite part of the space", since they are not essential for us and since the reader can easily establish them himself.

We construct ω as a product of isotopies, of the form

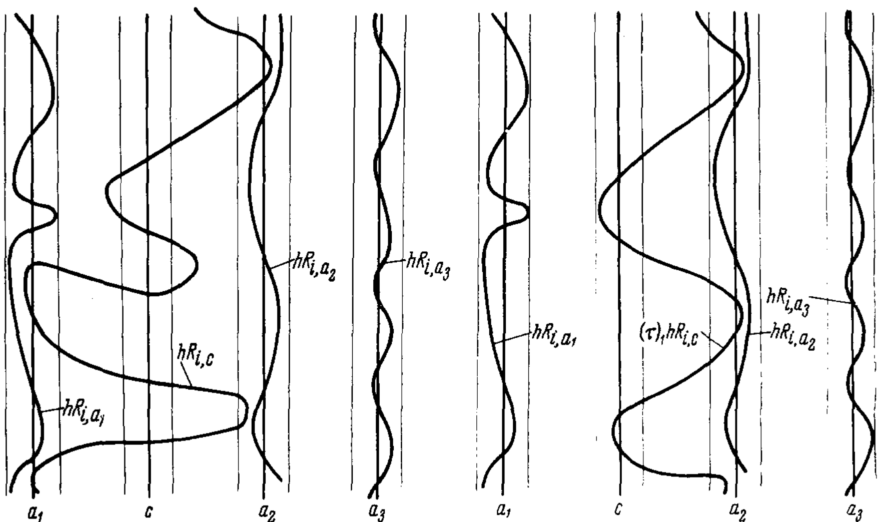
$$\omega = \sigma^{-1}\rho\sigma\tau,$$

where τ, ρ, and σ are constructed as follows:

Construction of τ. We choose a hyperplane R_{i,b_1} such that $hR_{i,b_1} \subset R_{i,c}^+$, where $a_2 - \epsilon < b_1 < a_2$, and we let $\bar{\tau}$ be an isotopy that is fixed outside $R_1(a_1, a_2)$, takes $R_{i,c}$ into R_{i,b_1}, and takes every line parallel to the axis ox_i. Then $\tau = h\bar{\tau}h^{-1}$.

Construction of σ. Choose a hyperplane R_{i,b_2} such that $hR_{i,b_2} \subset R_{i,a_3}^\epsilon$, where $b_2 < a_3$, and let $\bar{\sigma}$ be an isotopy which is fixed outside $R_i(b_1, a_3)$, takes R_{i,a_3} into R_{i,b_2}, and takes every line parallel to ox_i into itself. Then $\sigma = (\tau h)\bar{\sigma}(\tau h)^{-1}$.

Construction of ρ. Let ρ be an isotopy which takes each line parallel to ox_i into itself, is the

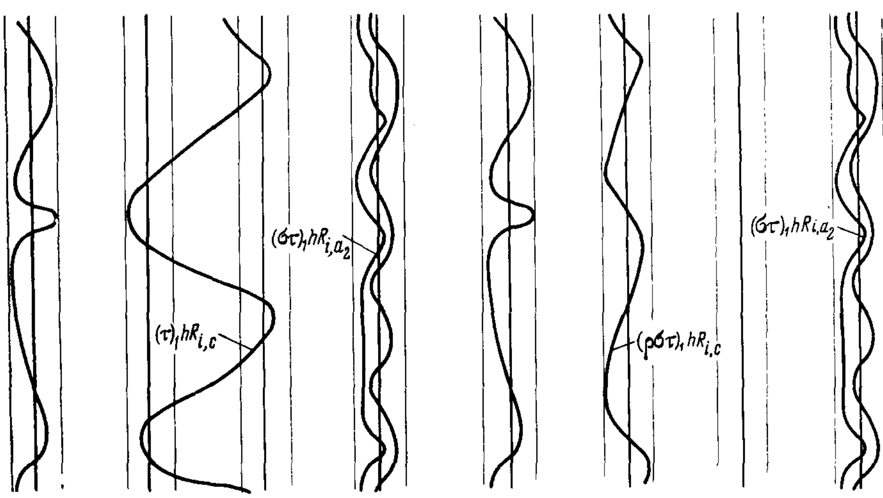


The homeomorphism h The homeomorphism $(\tau)_1 h$

Figure 1

identity outside $R_i(c-\epsilon,\ a_3-\epsilon)$, and takes $R_{i,a_2+\epsilon}$ into $R_{i,c+\epsilon}$.

One verifies immediately that if $\omega = \sigma^{-1}\rho\sigma\tau$ then ω possesses the required properties.



The homeomorphism $(\sigma\tau)_1 h$ The homeomorphism $(\rho\sigma\tau)_1 h$

Figure 2

Besides R_{i,a_i} and $R_{i,c}$, let hyperplanes $R_{i',a'_1}, \cdots, R_{i',a'_k}$, $a'_1 < a'_2 < \cdots < a'_k$, where $i' \neq i$, now be given. Suppose that for all j the ϵ-slabs R^{ϵ}_{i',a'_j} are pairwise disjoint and that $hR_{i',a'_j} \subset R^{\epsilon}_{i',a'_j}$.

How does our construction alter the situation in this second direction?

It is clear that the isotopies τ and σ take the image of each hyperplane $R_{i',a_j'}$ onto itself, and therefore these isotopies do not disturb the stated conditions; that is, $(\sigma\tau)_1 hR_{i',a_j'} \subset R^{\epsilon}_{i',a_j'}$. In just

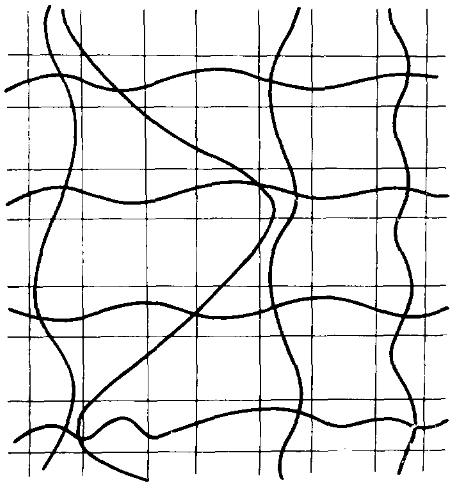


Figure 3

the same way, ρ maps each strip $R^{\epsilon}_{i',a_j'}$ onto itself, and so ρ does not disturb this condition either. However, ρ *displaces the image* $hR_{i',a_j'}$ *from its position in* R^n, and therefore when we apply σ^{-1} we are no longer justified in asserting that $(\sigma^{-1}\rho\sigma\tau)_1 hR_{i',a_j'}$ lies in $R^{\epsilon}_{i',a_j'}$. But since $(\rho\sigma\tau)_1 hR_{i',a_j'}$ still lies in $R^{\epsilon}_{i',a_j'}$ for all j, the homeomorphism $(\sigma^{-1})_1$ can take the points of this curved surface only into points that are taken by $(\sigma)_1$ into the strip $R^{\epsilon}_{i',a_j'}$. By the condition $hR_{i',a_j'} \subset R^{\epsilon}_{i',a_j'}$, satisfied for all j, we obviously have that $hR_{i'}(a'_{j-1}, a'_{j+1}) \supset R^{\epsilon}_{i',a_j'}$ for $j \neq 1$ and $j \neq k$, and since σ takes each hypersurface of the form $hR_{i',a_j'}$ onto itself we also have that $(\sigma)_1 hR_{i'}(a'_{j-1}, a'_{j+1}) \supset R^{\epsilon}_{i',a_j'}$. Moreover, obviously $(\sigma)_1 hR_{i'}(a'_{j-1}, a'_{j+1}) \subset R_{i'}(a'_{j-1} - \epsilon, a'_{j+1} + \epsilon)$.

On the other hand it is clear that $(\sigma^{-1}\rho\sigma\tau)_1 = e$ on hR^+_{i,a_2}, and so on $R^+_{i,a_2+\epsilon}$, and at the same time $(\sigma^{-1})_1 = (\sigma)_1 = e$ on $(\tau)_1 hR^-_{i,b_1}$, and so in any case on $R^-_{i,a_2-\epsilon}$. It follows that if for some j the image of $R_{i',a_j'}$ falls completely outside $R^{\epsilon}_{i',a_j'}$ under the homeomorphism $(\sigma^{-1}\rho\sigma\tau)_1 h$, then this happens, first, in the strip R^{ϵ}_{i,a_2}, and, second, for $j \neq 1$ or $j \neq k$ in the slab $R_{i'}(a'_{j-1} - \epsilon, a'_{j+1} + \epsilon)$. Thus we have the situation shown in Figure 4.

It is clear that if an additional hyperplane $R_{i,c'}$ between R_{i,a_1} and $R_{i,c}$ is given, with $R^{\epsilon}_{i,c'}$ not intersecting R^{ϵ}_{i,a_2} or $R^{\epsilon}_{i,c}$, then the whole argument can be repeated for $R_{i,c'}$, and a new isotopy ω' constructed which takes the image of $R_{i,c'}$ into $R^{\epsilon}_{i,c'}$ and which is the identity outside $hR_i(a_1, a_2)$, $(\omega')_1$ being the identity outside $hR_i(a, c)$, and the image of each hyperplane $R_{i',a_j'}$ remaining within the slab $R_{i'}(a'_{j-1} - \epsilon, a'_{j+1} + \epsilon)$. In fact, for any finite number of hyperplanes $R_{i,c_1}, R_{i,c_2}, \cdots, R_{i,c_l}$ between R_{i,a_1} and R_{i,a_2}, $a_1 < c_1 < \cdots < c_l < a_2$, we can construct an isotopy ω which takes hR_{i,c_j} into a narrow zone around R_{i,c_j} and simultaneously alters the situation in the orthogonal direction in the indicated manner. The thickness of the zones for R_{i,a_1} and R_{i,a_2} are no obstacle, because they can be shrunk as thin as required beforehand.

We now return to the situation of our theorem. We are given a homeomorphic δ-shift of I_2^n in R^n.

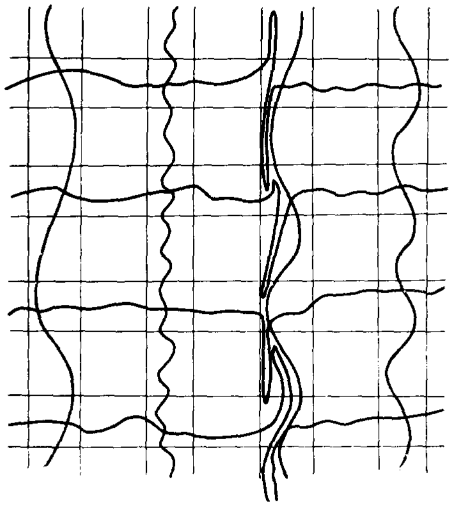


Figure 4

We wish to construct an isotopy $\mathrm{H}(g)$ which restores the image of I^n to its place and is the identity outside I_2^n.

For this purpose we consider the lattice formed by the $(n-1)$-dimensional cubes of the form $I_i(d;\, r_j)$, where i runs through the numbers from 1 to n, and d through some finite number of values depending on i and j, from $-r_j$ to r_j, and where $r_j = 1 + 1/2^j$. Moreover we can construct $\mathrm{H}(g)$ as a limit of isotopies, arranging at the jth step that the image of each cube of the above form lies in the ϵ_j-zone around this cube for some sufficiently small ϵ_j. The induction is started by the choice of δ. It is clear that if g is a δ-shift then the condition is satisfied for the given lattice if ϵ is chosen sufficiently small. As for the passage from j to $j+1$, we apply here the method described above, with the appropriate changes in connection with boundary conditions: after the jth step we require that the complement of the image of $I_{r_j}^n$ no longer moves.

At each step we construct the composition of the n isotopies associated with the respective n axes. As is seen from the above construction, when we obtain a refinement of the lattice in one direction we necessarily lose two-thirds of the "correct" $(n-1)$-cubes in each of the remaining directions (in order to be able to separate the images of the remaining cubes by hyperplanes and then to shrink them again to narrow bands). Therefore when we pass to the ith direction we have to multiply the number of "correct" cubes first by 2 and then by 3^{n-1}, in order to compensate for the losses incurred in this direction at the next $n-1$ steps, when we deal with the other directions. In actual fact there are also additional complications connected with the boundary conditions, but we need not go into these here.

3.3. We define some auxiliary isotopies; in the last analysis our construction will reduce to combinations of these.

By $\xi_i(d_1, d_2, d_3, d_4; r_1, r_2)$, where $d_1 < d_i < d_4$, $i = 2, 3$, we denote the isotopy which is the identity outside $\Pi_i(d_1, d_2; r_2)$, which on each segment parallel to ox_i with endpoints on $I_i(d_1; r_1)$ and $I_i(d_4; r_1)$ moves its points of intersection with $I_i(d_2; r_1)$ uniformly on t to the point of intersection with $I_i(d_3; r_1)$, which is linear on the complementary intervals of this segment, and which is linear on each segment of the ray extending from the center of the cube $I_i(d_2; r_1)$ between $\Pi_i(d_1, d_4; r_1)$ and $\Pi_i(d_1, d_4; r_2)$.

It is clear that the isotopy $\xi_i(d_1, d_2, d_3, d_4; r_1, r_2)$ firstly takes $I_i(d_2; r_1)$ into $I_i(d_3; r_1)$, and secondly takes each hyperplane parallel to ox_i into itself. We call an isotopy constructed in the above way an *isotopy of type* ξ.

Denote, also, the product $\xi_i(d - \epsilon_3, d - \epsilon_1, d - \epsilon_2, d; r_1, r_2)\xi_i(d, d + \epsilon_1, d + \epsilon_2, d + \epsilon_3; r_1, r_2)$ by $\nu_i(d; \epsilon_1, \epsilon_2, \epsilon_3; r_1, r_2)$. It is clear that the isotopy ν_i maps the "ϵ_1-slab" $\Pi_i(d - \epsilon_1, d + \epsilon_1; r_1)$ onto the "ϵ_2-slab" $\Pi_i(d - \epsilon_2, d + \epsilon_2; r_1)$ and is equal to e on $R^n \setminus \Pi_i(d - \epsilon_3, d + \epsilon_3; r_2)$.

3.4. We shall now prove a lemma in which the preliminary considerations of 3.2 are made precise, and which comprises the geometric kernel of the whole proof.

Lemma on the correction of homeomorphisms. *Let the axis* ox_i, $1 \le i \le n$, *be fixed, and suppose given*

a) *three numbers* $r_0, \bar{r}, r_1$ *such that* $1 < r_0 < \bar{r} < r_1 < 2$;

b) *for each* $i' \neq i$, $1 \le i' \le n$, $\kappa_{i'}$ *numbers* $d_{i',k}$, $1 \le k \le \kappa_{i'}$, *such that* $d_{i',1} = -r_0$, $d_{i',2} = -\bar{r}$, $d_{i',3} = -r_1 < d_{i',4} < \cdots < d_{i',\kappa_{i'}-3} < d_{i',\kappa_{i'}-2} = r_1$, $d_{i',\kappa_{i'}-1} = \bar{r}$ *and* $d_{i',\kappa_{i'}} = r_0$;

c) *numbers* d_0, d_1, d_2, d_3, *together with* λ *numbers* c_l, $1 \le l \le \lambda$, *such that* $-r_0 \le d_0 < d_1 < c_1 < c_2 < \cdots < c_\lambda < d_2 < d_3 \le r_0$.

Let $\epsilon > 0$ *be such that for each* $i' \neq i$ *the slabs* $R^\epsilon_{i',d_{i',k}}$ *are pairwise disjoint, and that the same is true for the set of slabs* R^ϵ_{i,c_l}, $1 \le l \le \lambda$, *and for the* R^ϵ_{i,d_m}, $m = 0, 1, 2, 3$.

Then, for each homeomorphic mapping $g\colon I^n_2 \to R^n$ *such that*

$$gI_{i'}(d; r_0) \subset I^\varepsilon_{i'}(d; r_0 + \varepsilon), \tag{*}$$

where $1 \le i' \le n$ *and* d *runs for* $i' \neq i$ *through the points* $d_{i',k}$ *and for* $i' = i$ *through* d_0, d_1, d_2, d_3, *there exists an isotopy* $\mathrm{X}(g)$ *of* R^n *such that* $(\mathrm{X})_0 = e$ *and the following conditions are satisfied.*

1 (X). X *depends continuously on* g;

2 (X) (1). $(\mathrm{X})_1 gI_i(c_l; r_1) \subset I^\epsilon_i(c_l; \bar{r} + \epsilon)$, $1 \le l \le \lambda$;

2 (X) (2). $(\mathrm{X})_1 g(I_{i'}(d; r_1) \cap R_i(d_1, d_2)) \subset \Pi_{i'}(d' - \epsilon, d'' + \epsilon; \bar{r} + \epsilon) \cap R_i(d_1 - \epsilon, d_2 + \epsilon)$, *where* $i' \neq i$, d *runs through the points* $d_{i',k}$ *in the interval* $[-r_1, r_1]$, *and* d', d'' *are the left and right neighbors of* d *among these points*;

3 (X) (1). $\mathrm{X} \equiv e$ *on* $R^n \setminus g\Pi_i(d_1, d_3; r_0)$;

3 (X) (2). $(\mathrm{X})_1 = e$ *on* $R^n \setminus g\Pi_i(d_1, d_2; r_0)$.

(In view of condition 3 (X) (1) it is immaterial in which topology we consider the space of isotopies of R^n.)

3.5. Before proving the lemma, we derive some consequences from condition (*) for the homeomorphism g:

$$g(\Pi_{i'}(d', d''; r) \cap R_i(d_1, d_2)) \subset \Pi_{i'}(d' - \varepsilon, d'' + \varepsilon; r + \varepsilon) \cap R_i(d_1 - \varepsilon, d_2 + \varepsilon), \tag{1}$$

where $i' \neq i$, and d', d'' run through the pairs of points $d_{i',k}$ in the interval $[-r, r]$, with $d' \leq d''$ and $r = r_1$ on $\bar{r}$.

Let $r = \bar{r}$, say. We first notice that the left side of (1) lies outside the image of $\Pi_{i'}(d'', r_0; r_0) \cap R_i(d_0, d_3)$. At the same time, by applying condition (*) to each face of this parallelepiped we see that the segment joining an arbitrary point κ of its boundary to its image gx lies outside $\Pi_{i'}(d'' + \epsilon, r_0 - \epsilon; r_0 - \epsilon) \cap R_i(d_0 + \epsilon, d_3 - \epsilon)$, and so this boundary can be taken into its image by a deformation outside the parallelepiped. Since the second parallelepiped lies in the first, we have

$$g(\Pi_{i'}(d'', r_0; r_0) \cap R_i(d_0, d_3)) \supset \Pi_{i'}(d'' + \varepsilon, r_0 - \varepsilon; r_0 - \varepsilon) \cap R_i(d_0 + \varepsilon, d_3 - \varepsilon).$$

From the above it follows that the region $g(\Pi_{i'}(d', d''; \bar{r}) \cap R_i(d_1, d_2))$ lies outside $\Pi_i(d'' + \epsilon, r_0 + \epsilon; r_0 - \epsilon) \cap R_i(d_0 + \epsilon, d_3 - \epsilon)$.

In exactly the same way, this region lies outside $\Pi_{i'}(-r_0 + \epsilon, d' - \epsilon; r_0 - \epsilon) \cap R_i(d_0 + \epsilon, d_3 - \epsilon)$.

It is proved similarly that it lies outside the parallelepipeds $\Pi_{i''}(\bar{r} + \epsilon, r_0 - \epsilon; r_0 - \epsilon) \cap R_i(d_0 + \epsilon, d_3 - \epsilon)$ and $\Pi_{i''}(-r_0 + \epsilon, -\bar{r} - \epsilon; r_0 - \epsilon) \cap R_i(d_0 + \epsilon, d_3 - \epsilon)$, and also outside the parallelepipeds $\Pi_i(d_2 + \epsilon; r_0 - \epsilon)$ and $\Pi_i(d_2 + \epsilon, d_3 - \epsilon; r_0 - \epsilon)$. In all, we obtain that this region cannot intersect the difference of parallelepipeds

$$\Pi_i(d_0 + \varepsilon, d_3 - \varepsilon; r_0 - \varepsilon) \setminus (\Pi_{i'}(d' - \varepsilon, d'' + \varepsilon; \bar{r} + \varepsilon) \cap R_i(d_1, d_2)) \tag{2}$$

or, in particular, the boundary of the former. But it intersects the first parallelepiped, and so is contained in it. In fact, the point $g(x)$, where the coordinates of x are $x_i = d_1$, $x_{i'} = d'$ and $x_{i''} = r_1$, where $i'' \neq i$ and $i'' \neq i'$, firstly belongs to this region and secondly belongs to $\Pi_i(d_0 + \epsilon, d_3 - \epsilon; r_0 - \epsilon)$, by condition (*) applied to all the coordinates of x.

From the above it follows that the region in question lies in the smaller parallelepiped of the difference (2), as was to be proved.

We observe that in the particular case when $d' = d'' = d$ we have

$$g(I_{i'}(d; r) \cap R_i(d_1, d_2)) \subset I^{\varepsilon}_{i'}(d; r + \varepsilon) \cap R_i(d_1 - \varepsilon, d_2 + \varepsilon), \tag{3}$$

where $r = r_1$ or $\bar{r}$, $i' \neq i$, and $d = d_{i',k}$.

It is similarly proved that

$$gI_i(d; r) \subset I^{\varepsilon}_i(d; r + \varepsilon), \tag{4}$$

where $r = r_1$ or $\bar{r}$, and $d = d_1$ or d_2, and also that

$$g\Pi_i(d_1, d_2; r) \subset \Pi_i(d_1 - \varepsilon, d_2 + \varepsilon; r + \varepsilon), \tag{5}$$

where $r = r_1$, $\bar{r}$, r_0.

3.6. We shall construct X as the composition of λ isotopies ω_l, $1 \leq l \leq \lambda$, which successively "correct" the images of the cubes $I_i(c_l; r_1)$ corresponding to the points c_l. We begin with the point c_λ; that is, $X = \omega_\lambda \circ \cdots \circ \omega_1$.

For the sake of uniformity put $c_{\lambda+1} = d_2$ and $c_{\lambda+2} = d_3$. Both these points are used in the construction of ω_λ, and the point d_2 also in the construction of $\omega_{\lambda-1}$. Denote by H the set $\bigcup_{i' \neq i;\, 1 \leq i \leq n} (R^\epsilon_{i', r_0} \cup R^\epsilon_{i', -r_0})$; then

$$H \cap I^n_{r_0-\varepsilon} = \Lambda. \tag{6}$$

3.7. We shall construct ω_l as the product of an isotopy $\bar{\omega}_l$ for which $(\bar{\omega}_l)_0 = e$, and the homeomorphism equal to e for $l = \lambda$ and $(\omega_{l+1})_1$ for $l < \lambda$.

For the isotopies ω_l we require the following conditions (given in a form convenient for induction) to be satisfied:

1 (ω). ω_l depends continuously on g;

2 (ω) (1a). $(\omega_l)_1 g I_i(c_{l'}; \bar{r}) \subset I^\epsilon_i(c_{l'}; \bar{r} + \epsilon)$, $l \leq l' \leq \lambda$;

2 (ω) (1b). $(\omega_l)_1 g I_i(c_{l'}; r_0) \subset R^+_{i, c_{l'}-\epsilon} \cup H$, $l \leq l' \leq \lambda$;

2 (ω) (2a). $(\omega_l)_1 g(I_{i'}(d; r_1) \cap R_i(d_1, d_2)) \subset (I^\epsilon_{i'}(d; r_1 + \epsilon) \cap R_i(d_1 - \epsilon, d_2 + \epsilon)) \cup (\Pi_{i'}(d' - \epsilon, d'' + \epsilon; \bar{r} + \epsilon) \cap \bigcup_{l'=l+1}^{\lambda+1} I^\epsilon_i(c_{l'}; \bar{r} + \epsilon))$, where i', d, d' and d'' are as in 2(X)(2);

2 (ω) (2b). $(\omega_l)_1 g(\Pi_{i'}(d', d''; \bar{r}) \cap R_i(d_1; c_l)) \subset \Pi_{i'}(d' - \epsilon, d'' + \epsilon; \bar{r} + \epsilon) \cap R_i(d_1 - \epsilon, c_l + \epsilon)$, where $i' \neq i$, $1 \leq i' \leq n$, $d' \leq d''$, and d', d'' run over the points $d_{i', k}$ of the interval $[-\bar{r}, \bar{r}]$;

2 (ω) (2c). $(\omega_l)_1 g(\Pi_{i'}(d', d''; \bar{r}) \cap R_i(d_1, d_2)) \subset (\Pi_{i'}(d' - \epsilon, d'' + \epsilon; \bar{r} + \epsilon) \cap R_i(d_1 - \epsilon, d_2 + \epsilon)) \cup (R^+_{i, c_{l+1}-\epsilon} \cup H)$, where i', d', and d'' are as in (2b);

3 (ω) (1). $\bar{\omega}_l \equiv e$ on $R^n \setminus (\omega_{l+1})_1 g \Pi_i(d_1, c_{l+2}; r_0)$;

3 (ω) (2). $(\bar{\omega}_l)_1 = e$ on $R^n \setminus (\omega_{l+1})_1 g \Pi_i(d_1, c_{l+1}; r_0)$.

3.8. Conditions (X) follow from conditions (ω). This is clear for 1(X) and 3(X). Condition 2(X)(1) follows from 2(ω)(1a) for $l = 1$, and 2(X)(2) from 2(ω)(2a) for $l = 1$, since, as one sees immediately, both terms on the right-hand side of the latter condition lie in each term of the right-hand side of condition 2(X)(2).

3.9. Arguing by induction, suppose that the isotopy ω_{l+1} has already been constructed, and so a homeomorphism $\tilde{g}_{l+1}$ is defined, equal to g in the case $l = \lambda$ and equal to $(\omega_{l+1})_1 g$ for $l < \lambda$. From conditions (ω) it obviously follows here that

1 $(\tilde{g})$. $\tilde{g}_{l+1}$ depends continuously on g;

2 $(\tilde{g})$. the same as 2(ω), but with $(\omega_l)_1 g$ replaced by $\tilde{g}_{l+1}$ and l by $l + 1$;

3 $(\tilde{g})$. $\tilde{g} = g$ on $R^n \setminus \Pi_i(d_1, d_2; r_0)$.

These conditions are satisfied for g in the case $l = \lambda$, and therefore we can start our induction. In fact, 1$(\tilde{g})$ and 3$(\tilde{g})$ are satisfied trivially; 2$(\tilde{g})$(1a) and (1b) follow from the fact that by hypothesis, according to $(*)$, $g I_i(d_2; r_0) \subset I^\epsilon_i(d_2; r_0 + \epsilon)$; 2$(\tilde{g})$(2a) follows from (3); and 2$(\tilde{g})$(2b) and (2c) both follow from (4).

Henceforth we drop the index l in the notation for the isotopies and the index $l + 1$ in the notation for the homeomorphism $\tilde{g}_{l+1}$.

Moreover, we carry out the argument for the case $l < \lambda$. The case $l = \lambda$ requires only minor alterations.

3.10. We derive some corollaries from condition $2(\tilde{g})$.

$$\tilde{g}\Pi_i(d_1,\ d_2;\ r_1)\subset\Pi_i(d_1-\varepsilon,\ d_2+\varepsilon;\ r_1+\varepsilon)\cup R^+_{i,c_{l+2}-\varepsilon}. \tag{7}$$

It is sufficient to prove that the boundary of the left-hand region is contained in the right-hand one. For the images of the faces $I_i(d_1;\ r_1)$ and $I_i(d_2;\ r_1)$ this follows from (4), since $\tilde{g}=g$ by $3(\tilde{g})$. If $i'\neq i$ then

$$\tilde{g}(I_{i'}(\pm r_1;\ r_1)\cap R_i(d_1,\ d_2))\subset(I^{\varepsilon}_{i'}(\pm r_1;\ r_1+\varepsilon)\cap R_i(d_1-\varepsilon,\ d_2+\varepsilon))\cup R^+_{i,c_{l+1}-\varepsilon}$$

by $2(\tilde{g})$ (2a), since $I_i^{\epsilon}(c_{i'};\ \bar{r}+\epsilon)\subset R^+_{i,c_{l+2}-\epsilon}$ for $l'\geq l+2$. Since obviously $I^{\epsilon}_{i'}(\pm r_1;\ r_1+\epsilon)\subset I^n_{r_1+\epsilon}$, we obtain the required conclusion.

$$\tilde{g}\Pi_i(d_0,\ d_2;\ \bar{r})\supset\Pi_i(d_1,\ c_{l+2}-\varepsilon;\ r_1+\varepsilon). \tag{8}$$

We observe that the boundary of the left-hand region does not intersect the interior of the right-hand one. For the faces $I_i(d_0;\ \bar{r})$ and $I_i(d_2;\ \bar{r})$ this follows from condition (*) for g, since $\tilde{g}=g$ on them, by $3(\tilde{g})$. For the remaining faces it follows from $2(\tilde{g})$ (2c) (for $d'=d''=\bar{r}$) and (7).

On the other hand, the two regions intersect; for example, they both contain $\tilde{g}I_i(c_{l+1};\ r_1)$. This is clear for the left-hand side, and for the right-hand side it follows from $2(\tilde{g})$(1a) and (7).

From the above two assertions it follows that the left-hand region contains the right-hand one; that is, (8) is proved.

$$\tilde{g}\Pi_i(d_1,\ c_{l+1};\ \bar{r})\subset\Pi_i(d_1-\varepsilon,\ c_{l+1}+\varepsilon;\ \bar{r}+\varepsilon). \tag{9}$$

This condition comes directly from $2(\tilde{g})$ (2b), upon putting $d'=-\bar{r},\ d''=\bar{r}$.

$$\tilde{g}\Pi_i(d_1,\ c_{l+2};\ r_0)\supset\Pi_i(d_1+\varepsilon,\ c_{l+2}-\varepsilon;\ r_0-\varepsilon). \tag{10}$$

The proof is similar to that of (8). In fact, that the boundary of the left-hand region does not intersect the interior of the right-hand one follows from (*) for the face $I_i(d_1;\ r_0)$, from $2(\tilde{g})$ (1b) and (6) for the face $I_i(c_{l+2};\ r_0)$ and, finally, from (*) and (3) for the faces $I_{i'}(\pm r_0;\ r_0)\cap R_i(d_1,\ c_{l+2})$. On the other hand, both regions contain $\tilde{g}I_i(c_{l+1};\ r)$ (see $2(\tilde{g})$ (1a).

$$\tilde{g}(\Pi_{i'}(d',\ d'';\ \bar{r})\cap R_i(d_1,\ c_{l+2}))\supset I^{\varepsilon}_{i'}(d;\ r_1+\varepsilon)\cap R_i(d_1+\varepsilon,\ c_{l+2}-\varepsilon), \tag{11}$$

where $i'\neq i$, d is one of the points $d_{i',k}$ in the interval $[-r_1,\ r_1]$, and $d',\ d''$ are the left and right neighbors among the points $d_{i',k}$.

Both sides of this inclusion contain $\tilde{g}(I_{i'}(d;\ r_1)\cap R_i(d_1,\ c_{l+1}))$: the left-hand side obviously, and the right by $2(\tilde{g})$ (2a) and (2b). Therefore it is again sufficient to prove that the boundary of the left-hand region does not intersect the interior of the right-hand one.

For the faces orthogonal to ox_i this follows from (*), $3(\tilde{g})$, and $2(\tilde{g})$ (1b).

For the faces of the form $I_{i''}(\pm\bar{r},\ \bar{r})\cap R_{i'}(d',\ d'')\cap R_i(d_1,\ c_{l+2})$, where $i'\neq i$ and $i''\neq i$, we have from $2(\tilde{g})$ (2c) that

$$\tilde{g}(I_{i''}(\pm\bar{r},\ \bar{r})\cap R_i(d_1,\ d_2))\subset(I^{\varepsilon}_{i''}(\pm\bar{r},\ \bar{r}+\varepsilon)\cap R_i(d_1-\varepsilon,\ d_2+\varepsilon))\cup R^+_{i,c_{l+2}-\varepsilon}\cup H,$$

and the right-hand side of this inclusion does not intersect the interior of the region standing on the right-hand side of (11).

Finally, we see that it follows from condition $2(\tilde{g})$ (2c) that the image of the face $I_{i'}(d'; \bar{r}) \cap R(d_1; c_{l+2})$ lies in

$$I_{i'}^{\varepsilon}(d'; \bar{r}+\varepsilon) \cap R_i(d_1-\varepsilon,\ d_2+\varepsilon) \cup R^{+}_{i,c_{l+2}-\varepsilon} \cup H,$$

and again

$$I_{i'}^{\varepsilon}(d'; \bar{r}+\varepsilon) \cap I_{i'}^{\varepsilon}(d; r_1+\varepsilon) = \Lambda, \quad R^{+}_{i,c_{l+2}-\varepsilon} \cap \operatorname{Int} R^{-}_{i,c_{l+2}-\varepsilon} = \Lambda,$$
$$H \cap I^{n}_{r_1+\varepsilon} = \Lambda \ \text{(see (6))}.$$

3.11. We pass to the construction of $\bar{\omega}$. We observe that by an arbitrarily small isotopy, independent of g, we can arrange for the weak inclusion in $2(\tilde{g})$ to be replaced by strong; that is, by inclusion in the interior. We shall assume this to have been done already.

We shall construct $\bar{\omega}$ as a product $\sigma^{-1}\rho\sigma\tau$, where τ, σ, and ρ are three isotopies, of which τ and σ are conjugate to isotopies of type ξ, and ρ itself has type ξ (see 3.4).

3.12. Construction of τ. Let $a_1 \geq c_l$ be the least number with the property that

$$\tilde{g} I_i(a_1; r_0) \subset R^{+}_{i,c_l-\varepsilon} \cup H. \tag{12}$$

This minimum exists, with

$$a_1 < c_{l+1}, \tag{13}$$

by $2(\tilde{g})$ (1b).

Let us prove that $a_1 = a_1(\tilde{g})$ depends continuously on $\tilde{g}$. To do this we first show that if a number $\bar{a}$ has property (12) then for any a' such that $\bar{a} < a' \leq c_{l+1}$

$$\tilde{g} I_i(a', r_0) \cap \Pi_i(d_1+\varepsilon,\ c_l-\varepsilon;\ r_0-\varepsilon) = \Lambda. \tag{14}$$

In fact, the image of the boundary of $\Pi_i(d_1, \bar{a}; r_0)$ obviously does not intersect $\operatorname{Int}\Pi_i(d_1+\epsilon, c_l-\epsilon; r_0-\epsilon)$ (by $(*)$, (12), and (6)), while this boundary may be deformed into its image outside the parallelepiped $\Pi_i(d_1+\epsilon, c_l-\epsilon; r_0-\epsilon)$. Indeed, the points of all the faces of $\Pi_i(d_1, \bar{a}; r_0)$, other than $I_i(\bar{a}; r_0)$, go into their images along the segments joining them, and the image of the latter face must first be deformed on H onto $R^{+}_{i,c_l-\epsilon}$, and then also deformed along segments.

From this we obtain that

$$\tilde{g}\Pi_i(d_1,\ \bar{a};\ r_0) \supset \Pi_i(d_1+\varepsilon,\ c_l-\varepsilon;\ r_0-\varepsilon). \tag{15}$$

On the other hand, it is clear that for $a' < \bar{a}$ we have

$$\tilde{g} I_i(a'; r_0) \cap \operatorname{Int}\tilde{g}\Pi_i(d_1,\ \bar{a};\ r_0) = \Lambda.$$

From this and (15) we obtain (14).

If now ϵ is a small number, then by the above it follows from the choice of a_1 that

$$\tilde{g} I_i(a+\varepsilon; r_0) \cap \Pi_i(d_1+\varepsilon,\ c_l-\varepsilon;\ r_0-\varepsilon) = \Lambda,$$

while

$$\tilde{g}I_i(a_1-\varepsilon;\ r_0)\cap \operatorname{Int}\Pi_i(d_1+\varepsilon;\ c_l-\varepsilon;\ r_0-\varepsilon)\neq\Lambda,$$

since otherwise $a_1-\epsilon$ would have property (12), and then a_1 would not be the least number with this property.

Both the conditions are preserved if $\tilde{g}$ is replaced by a sufficiently close homeomorphism. This proves the continuity of $a_1(\tilde{g})$.

Now let $a_2=a_2(\tilde{g})$ be the greatest number such that for all a', $a<a'<a_2$, we have

$$\tilde{g}(\Pi_i(d_1,\ d_2;\ r_0)\setminus I^n_{r_0-a'})\subset \operatorname{Int}H. \tag{16}$$

This time we shall not prove that a_2 depends continuously on $\tilde{g}$, but we shall show that $a_2(\tilde{g})$ is lower semicontinuous as well as, by the remark at the beginning of 3.11, strictly positive. To prove the lower semicontinuity we observe that if $0<a'<a_2(\tilde{g})$ then condition (16) is also satisfied for all homeomorphisms $g':\ I^n_2\to R^n$ sufficiently close to $\tilde{g}$, and therefore $a_2(g')>a'$, whence follows the semicontinuity of a_2.

By Baire's well-known theorem on the separation of semicontinuous functions by continuous ones (see [4]), we can find a continuous strictly positive function $\tilde{a}_2(\tilde{g})$ such that

$$\tilde{g}(\Pi_i(d_1,\ d_2;\ r_0)\setminus I^n_{r_0-\tilde{a}_2})\subset H. \tag{17}$$

Now let $\bar{\tau}=\xi_i(d_1,\ c_l,\ a_1,\ c_{l+1};\ r_2-\tilde{a}_2,\ r_0)$.

Since $d_1<c_l\le a_1<c_{l+1}$, $\bar{\tau}$ is correctly defined. We note that

$$\bar{\tau}I_i(c_l;\ r_0)\subset I_i(a_1;\ r_0)\cup(\Pi_i(d_1,\ d_2;\ r_0)\setminus I^n_{r_0-\tilde{a}_2}). \tag{18}$$

We put

$$\tau=\tilde{g}\bar{\tau}\tilde{g}^{-1}.$$

We remark immediately that from the construction of τ and from (18) it follows that

$$(\tau)_1\tilde{g}I_i(c_l;\ r_0)\subset\tilde{g}I_i(a_1;\ r_0)\cup\tilde{g}(\Pi_i(d_1,\ d_2;\ r_0)\setminus I^n_{r_0-\tilde{a}_2}), \tag{19}$$

and by the choice of a_1 (see (12)) and by (17),

$$(\tau)_1\tilde{g}I_i(c_l;\ r_0)\subset R^+_{i,c_l-\varepsilon}\cup H. \tag{20}$$

Moreover, obviously

$$(\tau)_1\tilde{g}I_i(c_l;\ r_0)\subset\tilde{g}\Pi_i(d_1,\ a_1;\ r_0). \tag{21}$$

3.13. We note the following properties of τ:

1 (τ). τ depends continuously on $\tilde{g}$;

2 (τ). the same as $2(\tilde{g})$, but with $\tilde{g}$ replaced by $(\tau)_1\tilde{g}$ and with the following addition to 1(a):

$$\tilde{g}I_i(c_l;\ \bar{r})\subset\Pi_i(c_l-\varepsilon,\ c_{l+1}+\varepsilon;\ \bar{r}+\varepsilon); \tag{22}$$

3 (τ). $\tau\equiv e$ on $R^n\setminus\tilde{g}\Pi_i(d_1,\ c_{l+1};\ r_0)$.

Property 1 (τ) follows from the fact that both the parameters a_1 and $\tilde{a}_2$ defining the isotopy $\bar{\tau}$ depend continuously on $\tilde{g}$, by their construction; and 3 (τ) is also obvious from the construction. To prove $2(\tau)$ and for later use we note that since $\bar{\tau}$ is an isotopy of type ξ, it takes every hyperplane parallel to ox_i into itself, and so τ takes the images of such hyperplanes under $\tilde{g}$ into themselves.

From these remarks and from $3(\tau)$ it follows that

$$(\tau)_1\tilde{g}I_{i'}(d;\ r_0) = \tilde{g}I_{i'}(d;\ r_0),\ \ i' \neq i, \tag{23}$$

$$(\tau_1)\tilde{g}I_{i'}(d;\ \bar{r}) = \tilde{g}I_{i'}(d;\ \bar{r}),\ \ i' \neq i, \tag{24}$$

$$(\tau)_1\tilde{g} = \tilde{g}\ \text{ on }\ I_{i'}(d;\ r_0) \cap R^+_{i,c_{l+1}},\ \ i' \neq i. \tag{25}$$

In turn it follows from these properties that $(\tau)_1$ takes the left sides of the inclusions in conditions $2(\tilde{g})$ (2) into themselves. The conditions $2(\tilde{g})$ (1) are not disturbed, since $(\tau)_1 = e$ on their left sides, by $3(\tau)$.

We note further that from (24) and (25) it follows that

$$(\tau)_1\tilde{g}\Pi_i(d_1,\ c_{l+1};\ \bar{r}) = \tilde{g}\Pi_i(d_1,\ c_{l+1};\ \bar{r}). \tag{26}$$

Finally we prove the addition (22) to condition (1a). From $3(\tau)$ it follows that

$$(\tau)_1\tilde{g}I_i(c_l;\bar{r}) \subset \tilde{g}\Pi_i(d_1,\ c_{l+1};\ \bar{r}). \tag{27}$$

From (9) it now follows that

$$(\tau)_1\tilde{g}I_i(c_l;\ \bar{r}) \subset \Pi_i(d_1 - \varepsilon,\ c_{l+1} + \varepsilon;\ \bar{r} + \varepsilon). \tag{28}$$

From (21) and (28) it follows that

$$(\tau)_1\tilde{g}I_i(c_l;\ \bar{r}) \subset (R^+_{i,c_l-\varepsilon} \cup H) \cap \Pi_i(d_1 - \varepsilon,\ c_{l+1} + \varepsilon;\ \bar{r} + \varepsilon),$$

which in view of (6) is $\Pi_i(c_l - \epsilon,\ c_{l+1} + \epsilon;\ \bar{r} + \epsilon)$, and thus (22) is proved.

We note for future use one further property of τ, which follows from the fact that $\bar{\tau}$ is constructed as an isotopy of ξ, and from $3(\tau)$ ($r = r_0$, $\bar{r}$, or r_1):

$$(\tau)_1\tilde{g}\Pi_i(d_1,\ d_2;\ r) = \tilde{g}\Pi_i(d_1,\ d_2;\ r). \tag{29}$$

3.14. Construction of σ. Let $b_1 = b_1(\tilde{g})$ be the least number such that

$$b_1 \geqslant \max(c_l,\ a_1), \tag{30}$$

$$(\tau)_1\tilde{g}I_i(b_1;\ r_0) \subset R^+_{i,c_{l+1}-\varepsilon} \cup H. \tag{31}$$

Such a minimum exists, with $b_1 < c_{l+1}$, by $3(\tau)$ and the assumption of strict inclusion in 3.12.

Like a_1, b_1 depends continuously on $\tilde{g}$. We note some properties of b_1:

$$(\tau)_1\tilde{g}I_i(c_l;\ r_0) \subset \tilde{g}\Pi_i(d_1,\ b_1;\ r_0) \tag{32}$$

(from (30) and (21));

$$(\tau)_1\,\tilde{g}I_{i'}\ (d;\bar{r}) \cap R_i(d_1, b_1)) \subset \tilde{g}\,(I_{i'}\ (d;\bar{r}) \cap R_i(d_1, c_{l+1})) \tag{33}$$

(from (32), (24), and (26));

$$\tau_1\widetilde{g}\Pi_i(d_1,\ b_1;\ \bar{r})\subset\widetilde{g}\,\Pi_i(d_1,\ c_{l+1},\ \bar{r});\tag{34}$$

$$(\tau)_1\widetilde{g}\Pi_i(b_1,\ c_{l+2};\ r_0)\subset R^+_{i,c_{l+1}-\varepsilon}\cup H.\tag{35}$$

Indeed, the boundary of the left-hand region lies in the right-hand one.

$$(\tau)_1\widetilde{g}\Pi_i(b_1,\ c_{l+1};\ \bar{r})\subset I^\varepsilon_l(c_{l+1};\ \bar{r}+\varepsilon).\tag{36}$$

Since $b_1\geq c_l>d_1$, it follows from (26) and (9) that

$$(\tau)_1\widetilde{g}\Pi_i(b_1,\ c_{l+1};\ \bar{r})\subset\Pi_i(d_1-\varepsilon;\ c_{l+1}+\varepsilon;\bar{r}+\varepsilon).$$

On the other hand, from (34) we obtain that

$$(\tau)_1\widetilde{g}\Pi_i(b_1,\ c_{l+1};\ \bar{r})\subset R^+_{i,c_{l+1}-\varepsilon}\cup H.$$

Since $H\cap I^b_{\bar{r}+\epsilon}=\Lambda$ (see (6)), we obtain (35).

Now let $b_2=b_2(\widetilde{g})$ be the least number such that

$$(\tau)_1\widetilde{g}I_l(b_2;\ r_0)\subset R^+_{i,c_{l+2}-\varepsilon}\cup H.\tag{37}$$

Again the minimum exists, with

$$c_{l+1}<b_2<c_{l+2}\tag{38}$$

and with b_2 depending continuously on $\widetilde{g}$.

We observe that

$$(\tau)_1\widetilde{g}\Pi_l(b_2,\ c_{l+2};\ r_0)\subset R^+_{i,c_{l+2}-\varepsilon}\cup H,\tag{39}$$

which proves (34).

Now let $\bar{\sigma}=\xi_i(b_1,\ c_{l+1},\ b_2,\ c_{l+2};\ r,\ r+\epsilon)$. Since $b_1<c_{l+1}<b_2<c_{l+2}$, $\bar{\sigma}$ is correctly defined. Put

$$\sigma=((\tau)_1\widetilde{g})\,\bar{\sigma}\,((\tau)_1g)^{-1}.$$

Then, by the construction,

$$(\sigma\tau)_1\,\widetilde{g}I_i\,(c_{l+1};\ r_0)=(\tau)_1\,\widetilde{g}I_i\,(b_2;\ r_0).\tag{40}$$

3.15. We note these properties of σ:

1 (σ). σ depends continuously on $\widetilde{g}$;

2 (σ) (1a). $(\sigma\tau)_1\widetilde{g}I_i(c_{l+1};\ r_0)\subset R^+_{i,c\,l+2-\epsilon}\cup H$;

2 (σ) (1b). $(\sigma\tau)_1\widetilde{g}=(\tau)_1g$ on $I_i(c_l;\ r_0)$;

2 (σ) (2). The same as 2($\widetilde{g}$) (2), with $\widetilde{g}$ replaced by $(\sigma\tau)_1\widetilde{g}$, and also c_{l+1} by b_1 in (2b).

3 (σ). $\sigma\equiv e$ on $R^n\setminus(\tau)_1\widetilde{g}\,\Pi_i(b_1,\ c_{l+2};\ r_0+\epsilon)$.

Properties 1(σ), 2(σ) (2), and 3(σ) can be verified in just the same way as the analogous properties for τ. Property 2(σ) (1b) follows from 3(σ) and (30). Finally, 2(σ) (1a) follows from (40) and (37).

We give some further properties of σ that are needed later:

$$(\sigma)_t\,(\tau)_1\,\widetilde{g}\,\Pi_i\,(d_1,\ d_3;\ r)=\widetilde{g}\,\Pi_i\,(d_1,\ d_3;\ r)\tag{41}$$

for all $t \in [0, 1]$ and for $r = r_1, \bar{r}, r_0$. This follows from (29), $3(\sigma)$, and the fact that $\bar{\sigma}$ is an isotopy of type ξ.

$$(\sigma\tau)_1 \tilde{g}\, \Pi_i (c_{l+1}, c_{l+2}; r_1) = (\tau)_1 \tilde{g}\, \Pi_i (b_2, c_{l+2}; r). \tag{42}$$

This follows from the construction.

From $3(\sigma)$ and (41) for $r = r_0$ it follows that

$$S(\sigma) \cap \tilde{g}\, \Pi_i (d_1, d_3; r_0) = (\tau)_1 \tilde{g}\, \Pi_i (b_1, c_{l+2}; r_0) \subset H \cup R^+_{i, c_{l+1}-\varepsilon}$$

(see (35)), where $S(\sigma)$ is the support of σ.

Hence it obviously follows that

$$(\sigma)_1 (R^+_{i, c_l-\varepsilon} \cup H) = R^+_{i, c_l-\varepsilon} \cup H, \tag{43}$$

$$(\sigma)_1 = e \text{ on } (\tau)_1 \tilde{g}\, \Pi_{i'} (d', d'', \bar{r}) \cap R_i (d_1, b_1),\ i' \neq i, \tag{44}$$

$$(\sigma)_1 = e \text{ on } I^\varepsilon_i (c_l; \bar{r} + \varepsilon), \tag{45}$$

$$(\sigma)_1 = e \text{ on } (\tau)_1 \tilde{g}\, \Pi_i (d_1, b_1; \bar{r}). \tag{46}$$

Finally, since τ and σ also take the image under $\tilde{g}$ of each hyperplane parallel to ox_i into itself, by $3(\sigma)$ and $3(\tau)$ we obtain that

$$\begin{aligned}(\sigma)_1 \tilde{g} (\Pi_{i'} (d', d''; \bar{r}) \cap R^-_{i, c_{l+2}}) &= (\sigma\tau)_1 \tilde{g} (\Pi_{i'} (d', d''; \bar{r}) \cap R^-_{i, c_{l+2}}) \\ &= \tilde{g} (\Pi_{i'} (d', d''; \bar{r}) \cap R^-_{i, c_{l+2}}),\end{aligned} \tag{47}$$

where $i' \neq i$, and $d', d'' \in \{d_{i', k}\}$, $d' \le d''$.

3.16. Construction of ρ. Put

$$\rho = \xi_i (c_l - \varepsilon, c_{l+1} + \varepsilon, c_l + \varepsilon, c_{l+2} - \varepsilon; \bar{r} + \varepsilon, r_0 - \varepsilon).$$

We observe that

$$(\rho)_1 \Pi_i (c_l - \varepsilon, c_{l+1} + \varepsilon; \bar{r} + \varepsilon) = I^\varepsilon_i (c_l; \bar{r} + \varepsilon). \tag{48}$$

3.17. We note these properties of ρ:

$1(\rho)$. ρ is independent of $\tilde{g}$;

$2(\rho)$ (1a). $(\rho\sigma\tau)_1 \tilde{g} I_i (c_l; \bar{r}) \subset I^\epsilon_i (c_l; \bar{r} + \epsilon)$;

$2(\rho)$ (1b). $(\rho\sigma\tau)_1 \tilde{g} I_i (c_l; r_0) \subset R^+_{i, c_l-\epsilon} \cup H$;

$2(\rho)$ (2a). $(\rho\sigma\tau)_1 \tilde{g} (I_{i'} (d; r_1) \cap R_i (d_1, d_3)) \subset (R_i (d_1 - \epsilon, d_2 + \epsilon) \cap I^\epsilon_{i'} (d; r_1 + \epsilon)) \cup$ $(\bigcup_{l'=l+2}^{\lambda+1} I^\epsilon_i (c_{l'}; \bar{r} + \epsilon) \cap \Pi_{i'} (d' - \epsilon, d'' + \epsilon; \bar{r} + \epsilon))$, where i', d, d', d'' are as in $2(\tilde{g})$ (2a);

$2(\rho)$ (2b). $(\rho\sigma\tau)_1 \tilde{g} (\Pi_{i'} (d', d''; \bar{r}) \cap R_i (d_1, c_l)) \subset \Pi_{i'} (d' - \epsilon, d'' + \epsilon; \bar{r} + \epsilon) \cap R_i (d_1 - \epsilon, c_l + \epsilon)$, where $d' \le d''$ and i' are as in $2(\tilde{g})$ (2b);

$2(\rho)$ (2c). $(\rho\sigma\tau)_1 \tilde{g} (\Pi_{i'} (d', d''; \bar{r}) \cap R_i (d_1, d_2)) \subset \Pi_{i'} (d' - \epsilon, d'' + \epsilon; \bar{r} + \epsilon) \cup$ $R_i (d_1; c_{l+2} - \epsilon) \cup H$, where d', d'', and i' are as in $2(\tilde{g})$ (2c);

$3(\rho)$. $\rho \equiv e$ on $R^n \setminus \Pi_i(c_l - \epsilon, c_{l+2} - \epsilon; r_0 - \epsilon)$.

Properties $1(\rho)$ and $3(\rho)$ follow immediately from the fact that ρ is of type ξ. Property $2(\rho)$ (1a) follows from (22), $2(\sigma)$ (1b), and (48), and $2(\rho)$ (1b) from (21), $2(\sigma)$ (1b), and the fact that $S(\rho) \subset R^+_{i, c_l - \epsilon}$ by $3(\rho)$, where $S(\rho)$ is the support of ρ.

Condition $2(\rho)$ (2a) follows from $2(\sigma)$ (2a), $2(\tilde{g})$ (2a), and the fact that ρ takes the right-hand side of the latter condition into itself, since $\rho \equiv e$ on $R^+_{i, c_{l+2} - \epsilon}$ (by $3(\rho)$), and since ρ takes $I^\epsilon_{i'}(d; r_1 + \epsilon)$ into itself (being of type ξ).

Condition $2(\rho)$ (2b) is obtained as follows:

$$\begin{aligned}
&(\rho\sigma\tau)_1 \tilde{g}(\Pi_{i'}(d', d''; \bar{r}) \cap R_i(d_1, c_l)) \\
&\subset (\rho\sigma\tau)_1 \tilde{g}(\Pi_{i'}(d', d''; \bar{r}) \cap R_i(d_1, b_1)) && \text{(see (30))} \\
&= (\rho\tau)_1 \tilde{g}(\Pi_{i'}(d', d''; \bar{r}) \cap R_i(d_1, b_1)) && \text{(see (44))} \\
&\subset (\rho)_1 \tilde{g}(\Pi_{i'}(d', d''; \bar{r}) \cap R_i(d_1, c_{l+1})) && \text{(see (34))} \\
&\subset (\rho)_1 (\Pi_{i'}(d' - \varepsilon, d'' + \varepsilon; \bar{r} + \varepsilon) \cap R_i(d_1 - \varepsilon, c_{l+1} + \varepsilon)) && \text{(see } 2(\tilde{g})\text{(2b))} \\
&= \Pi_{i'}(d' - \varepsilon, d'' + \varepsilon; \bar{r} + \varepsilon) \cap R_i(d_1 - \varepsilon, c_l + \varepsilon),
\end{aligned}$$

since ρ maps the first term into itself, and $R_i(d_1 - \epsilon, c_{l+1} + \epsilon) \cap I^n_{\bar{r}+\epsilon}$ into $R_i(d_1 - \epsilon, c_{l+\epsilon}) \cap I^n_{\bar{r}+\epsilon}$ by (48) and $3(\rho)$.

Finally, by $2(\sigma)$ (2c) and $2(\tilde{g})$ (2c),

$$\begin{aligned}
&(\rho\sigma\tau)_1 \tilde{g}(\Pi_{i'}(d', d''; \bar{r}) \cap R_i(d_1, d_3)) \\
&\subset (\rho)_1 (\Pi_{i'}(d' - \varepsilon, d'' + \varepsilon; \bar{r} + \varepsilon) \cup R_i(c_{l+2} - \varepsilon, d_2) \cup H),
\end{aligned}$$

and $(\rho)_1$ takes each term into itself, the first because ρ is of type ξ, and the second because $S(\rho)$ does not intersect it (see $3(\rho)$). Thus $2(\rho)$ (2c) is also proved.

We note some further properties of ρ:

$$(\rho)_1 \Pi_i(-r_1 - \varepsilon, c_{l+2} - \varepsilon; r_1 + \varepsilon) = \Pi_i(-r_1 - \varepsilon, c_{l+2} - \varepsilon; r_1 + \varepsilon); \tag{49}$$

$$(\rho)_1 R^-_{i, c_{l+2} - \varepsilon} = R^-_{i, c_{l+2} - \varepsilon}; \tag{50}$$

$$(\rho)_1 = e \text{ on } R^+_{i, c_{l+2} - \varepsilon} \cup H; \tag{51}$$

$$(\rho)_1 = e \text{ on } (\tau)_1 \tilde{g}\, \Pi_i(b_2, r_0; r_0); \tag{52}$$

$$(\rho)_1 = e \text{ on } (\sigma\tau)_1 \tilde{g}\, \Pi_i(c_{l+1}, c_{l+2}; r_0). \tag{53}$$

Property (49) follows from the fact that on the one hand $R_i(-r_i - \epsilon, c_{l+2} + \epsilon)$ contains the support of ρ, and on the other hand $(\rho)_1$ maps each hyperplane parallel to ox_i into itself. Properties (50) and (51) follow from the fact that the support of ρ lies in $R^-_{i, c_{l+2} - \epsilon}$ and does not intersect H, since $H \cap I^n_{r_0 - \epsilon} = \Lambda$ (see (6)). Property (52) follows from (51) and (39), and (53) from (52) and (42).

3.18. It remains to verify that the isotopy $\bar{\omega}$ possesses all the properties (ω). Condition 1 follows immediately from the corresponding properties of τ, σ, and ρ. Of these three isotopies only σ is not the identity on $R^n\setminus\tilde{g}\Pi_i(d_1, d_3; r_0)$, but by (41) $(\sigma)_t\tilde{g}\Pi_i(d_1, d_3; r_0) = \tilde{g}\Pi_i(d_1, d_3; r_0)$ for all t, and so $\sigma^{-1}\rho\sigma\tau = \sigma^{-1}\sigma \equiv e$ on $R^n\setminus\tilde{g}\Pi_i(d_1, d_3; r_0)$. Moreover τ is the identity on $\tilde{g}\Pi_i(c_{l+2}, d_3; r_0)$ by $3(\tau)$, σ is by $3(\sigma)$ and (41), and ρ is by $3(\rho)$ and (10). Thus $\bar{\omega} \equiv e$ on $R^n\setminus\tilde{g}\Pi_i(d_1, c_{l+2}; r_0)$, and $3(\omega)$ (1) has been verified.

Further, $(\tau)_1 = e$ on $\tilde{g}\Pi_i(c_{l+1}, c_{l+2}; r_0)$ by $3(\tau)$, and $(\rho)_1 = e$ on $(\sigma\tau)_1\tilde{g}\Pi_i(c_{l+1}, c_{l+2}; r_0)$ by (53). Thus $(\sigma^{-1}\rho\sigma\tau)_1 = (\sigma^{-1}\sigma\tau)_1 = (\sigma^{-1}\sigma)_1 = e$ on $\tilde{g}\Pi_i(c_{l+1}, c_{l+2}; r_0)$, and together with $3(\omega)$ (1) this gives $3(\omega)$ (2).

Let us verify conditions $2(\omega)$ (1). By $3(\omega)$ (2), condition (1a) need be proved only for $l' = l$, since for the remaining values of $l' \geq l$ it follows from the induction hypothesis, $2(\tilde{g})$ (1a) and $3(\omega)$ (2).

From $2(\rho)$ (1a) and (45) we have

$$(\sigma^{-1}\rho\sigma\tau)_1\,\tilde{g}I_i(c_l; \bar{r}) \subset (\sigma^{-1})_1\, I_i^{\varepsilon}(c_l; \bar{r}+\varepsilon) = I_i^{\varepsilon}(c_l; \bar{r}+\varepsilon).$$

Let us prove (1b). Again we need consider only the case $l' = l$. By $2(\rho)$ (1b) and (43) we have

$$(\sigma^{-1}\rho\sigma\tau)_1\tilde{g}I_i(c_l; r_0) \subset (\sigma^{-1})_1\,(R^+_{i,\,c_l-\varepsilon} \cup H) = R^+_{i,c_l-\varepsilon} \cup H,$$

which gives $2(\omega)$ (1b).

We pass to the proof of property $2(\omega)$ (2), beginning with (2c).

Let the point $x \in \tilde{g}(\Pi_i(d', d''; \bar{r}) \cap \Pi_i(d_1, c_{l+2}; r_0))$. If $(\rho\sigma\tau)_1 x \in (\tau)_1\tilde{g}\Pi_i(b_1, c_{l+2}; r_0)$, then in view of $3(\sigma)$ and (41) we also have $(\sigma^{-1}\rho\sigma\tau)_1 x \in (\tau)_1\tilde{g}\Pi_i(b_1, c_{l+2}; r_0)$, and $(\bar{\omega})_1 x \in R^+_{i,c_{l+1}-\epsilon} \cup H$ by (35).

If $(\rho\sigma\tau)_1 x \in (\tau)_1\tilde{g}\Pi_i(d_1, b_1; r_0)$ then, again by $3(\sigma)$, $(\sigma)_1((\rho\sigma\tau)_1 x) = (\rho\sigma\tau)_1 x$, and thus $(\bar{\omega})_1 x = (\rho\sigma\tau)_1 x$, so that $2(\rho)$ (2c) applies.

If $x \in \tilde{g}(\Pi_i(d', d''; \bar{r})\setminus\Pi_i(d_1, c_{l+2}; r_0))$ then $(\bar{\omega})_1 x = x$ by $3(\omega)$, and we apply $2(\tilde{g})$ (2c).

We now prove (2b). We have

$$\begin{aligned}(\sigma^{-1}\rho\sigma\tau)_1\tilde{g}\,(\Pi_{i'}(d', d''; \bar{r}) \cap R_i(d_1, c_l))\\ \subset (\sigma^{-1})_1(\Pi_{i'}(d'-\varepsilon, d''+\varepsilon; \bar{r}+\varepsilon) \cap R_i(d_1-\varepsilon, c_l+\varepsilon))\\ = \Pi_{i'}(d'-\varepsilon, d''+\varepsilon; \bar{r}+\varepsilon) \cap R_i(d_1-\varepsilon, c_l+\varepsilon),\end{aligned}$$

by $2(\rho)$ (2b) and the fact that $S(\sigma) \cap \tilde{g}\Pi_i(d_1, d_3; r_0) \subset (R^n\setminus I^n_{r_0-\epsilon}) \cup R^+_{i,c_{l+1}-\epsilon}$, where $S(\sigma)$ is the support of σ (see $3(\sigma)$ and (35)), and so $S(\sigma) \cap \tilde{g}\Pi_i(d_1, d_3; r_0)$ does not intersect $\Pi_{i'}(d'-\epsilon, d''+\epsilon; \bar{r}+\epsilon) \cap R_i(d_1-\epsilon; c_l+\epsilon)$ (since $\Pi_{i'}(d-\epsilon, d''+\epsilon; \bar{r}+\epsilon) \subset I^n_{\bar{r}+\epsilon} \subset I^n_{r_0-\epsilon}$ and $R^-_{i,c_l+\epsilon} \cap R^+_{i,c_{l+1}-\epsilon} = \Lambda$).

Finally we prove (2a). First we show that

$$(\bar{\omega})_1\,\tilde{g}\Pi_i(d_1, d_2; r_1) \subset \tilde{g}\Pi_i(d_1, d_2; \bar{r}). \tag{54}$$

Indeed,

$$(\rho\sigma\tau)_1\,\tilde{g}\Pi_i(d_1, d_2; r_1) = (\rho)_1\,\tilde{g}\Pi_i(d_1, d_2; r_1)$$

$$\subset (\rho)_1\,\Pi_i(d_1 - \varepsilon, d_2 + \varepsilon; r_1 + \varepsilon) \cup R^+_{i, c_{l+2}-\varepsilon} \qquad \text{(see (29), (41) and (7))}$$

$$= (\rho)_1\,\Pi_i(d_1 - \varepsilon, c_{l+2} - \varepsilon;\ r_1 + \varepsilon) \cup (\rho)_1\,R^+_{i, c_{l+2}-\varepsilon}$$

$$= \Pi_i(d_1 - \varepsilon, c_{l+2} - \varepsilon; r_1 + \varepsilon) \cup R^+_{i, c_{l+2}-\varepsilon} \qquad \text{(see (49) and (50))}$$

$$\subset \tilde{g}\,\Pi_i(d_0, d_2; \bar{r}) \cup R^+_{i, c_{l+2}-\varepsilon}. \qquad \text{(see (8))}$$

Take a point x in $\tilde{g}\Pi_i(d_1, d_2; r_1)$. If $(\rho\sigma\tau)_1 x \in R^+_{i, c_{l+2}-\epsilon}$ then, since $(\rho)_1 = e$ on $R^+_{i, c_{l+2}-\epsilon}$ (see $3(\rho)$), we have $(\rho\sigma\tau)_1 x = (\sigma\tau)_1 x$, and so $(\bar{\omega})_1 x = (\tau)_1 x$, and (29) can be applied. If $(\rho\sigma\tau)_1 x \in \tilde{g}\Pi_i(d_0, d_2; \bar{r})$ then $(\rho\sigma\tau)_1 x \in \tilde{g}\Pi_i(d_1, d_2; \bar{r})$ by $3(\omega)$, and it follows from (41) that we also have $(\bar{\omega})_1 x \in \tilde{g}\Pi_i(d_1, d_2; \bar{r})$.

It follows from (54) that $(\bar{\omega})_1\tilde{g}(I_{i'}(d; r_1) \cap R_i(d_1, d_2)) \subset \tilde{g}\Pi_i(d_1, d_2; \bar{r}) = (\tau)_1\tilde{g}\Pi_i(d_1, d_2; \bar{r})$ (see (29)). Now take a point $x \in \tilde{g}(I_{i'}(d; r_1) \cap R_i(d, d_2))$. If $(\bar{\omega})_1 x \in (\tau)_1\tilde{g}\Pi_i(d_1, b_1; \bar{r})$, then $(\sigma)_1((\bar{\omega})_1 x) = (\bar{\omega})_1 x$ by (46), and so $(\bar{\omega})_1 x = (\rho\sigma\tau)_1 x$, and $2(\omega)$ (2a) follows from $2(\rho)$ (2a). If $(\bar{\omega})_1 x \in (\tau)_1\tilde{g}\Pi_i(c_{l+1}, d_2; \bar{r})$ then $(\rho\sigma\tau)_1 x \in (\sigma\tau)_1\tilde{g}\Pi_i(c_{l+1}, d_2; \bar{r}) = (\tau)_1\tilde{g}\Pi_i(l_2, d_2; \bar{r})$ by (42), and $(\rho)_1 = e$ on $(\tau)_1\tilde{g}\Pi_i(b_2, d_2; \bar{r})$, by (52). Thus $(\rho\sigma\tau)_1 x = (\sigma\tau)_1 x$, and then $(\bar{\omega})_1 x = (\tau)_1 x$, and again $2(\tau)$ (2a) can be applied.

From what we have proved and (54) it follows that it is sufficient to consider the case when (see (36))

$$(\bar{\omega})_1\, x \in (\tau)_1\,\tilde{g}\Pi_i(b_1, c_{l+1}; \bar{r}) \subset I^{\varepsilon}_i(c_{l+1}; \bar{r} + \varepsilon). \tag{55}$$

On the other hand, by $2(\rho)$ (2a)

$$(\rho\sigma\tau)_1\,\tilde{g}(I_{i'}(d; r_1) \cap R_i(d_1, d_2))$$

$$\subset I^{\varepsilon}_{i'}(d; r_1 + \varepsilon) \cup (\Pi_{i'}(d' - \varepsilon, d'' + \varepsilon; \bar{r} + \varepsilon) \cap R^+_{i, c_{l+2}-\varepsilon}).$$

Again take $x \in \tilde{g}(I_{i'}(d; r_1) \cap R_i(d_1, d_2))$. Since $(\rho)_1 = e$ on $R^+_{i, c_{l+2}-\epsilon}$, if $(\rho\sigma\tau)_1 x$ lies in the second term then $(\rho\sigma\tau)_1 x = (\sigma\tau)_1 x$, $(\bar{\omega})_1 x = (\tau)_1 x$, and $2(\tau)$ (2a) applies. Since $I^{\epsilon}_{i'}(d; r_1 + \epsilon) \subset \Pi_{i''}(d' - \epsilon, d'' + \epsilon; \bar{r} + \epsilon)$, it is sufficient to consider the case when

$$(\rho\sigma\tau)_1\, x \in I^{\varepsilon}_{i'}(d; r_1 + \varepsilon) \cap R^-_{i, c_{l+2}-\varepsilon}$$

$$\subset \tilde{g}(\Pi_{i'}(d', d''; \bar{r}) \cap R_i(d_1, c_{l+2})) \qquad \text{(see (11))}$$

$$= (\sigma)_1\,\tilde{g}(\Pi_{i'}(d', d''; \bar{r}) \cap R_i(d_1, c_{l+2})) \qquad \text{(see (47))}$$

$$\subset (\sigma)_1\,(\Pi_{i'}(d' - \varepsilon, d'' + \varepsilon; \bar{r} + \varepsilon) \cup R^+_{i, c_{l+2}-\varepsilon} \cup H). \qquad \text{(see } 2(\tilde{g})\text{ (2c))}$$

In this case we have

$$(\bar{\omega})_1 x \subset \Pi_{i'}(d' - \varepsilon, d'' + \varepsilon; \bar{r} + \varepsilon) \cup R^+_{i, c_{l+2}-\varepsilon} \cup H. \tag{56}$$

Thus by (55) and (56), noting that $I_i^\epsilon(c_{l+1}; \bar{r} + \epsilon) \cap (R^+_{i,c_{l+2}-\epsilon} \cup H) = \Lambda$, we obtain that in the remaining case

$$(\bar{\omega})_1 x \in I_i^\varepsilon(c_{l+1}; \bar{r} + \varepsilon) \cap \Pi_{i'}(d' - \varepsilon, d'' + \varepsilon; \bar{r} + \varepsilon),$$

but this is one of the terms on the right-hand side of $2(\omega)$ (2a), and so this property has been verified.

The proof of the lemma is complete.

3.19. We observe in this subsection that our construction for the isotopy X enables the following two additional assertions to be made.

4(X). If $\eta = \max(d_3 - d_1, d_{i',k} - d_{i',k-1})$, where i' runs through all values from 1 to n except i, and k runs, for each $i' \neq i$, through all values from 2 to $\kappa_{i'}$, then ω is an $f(\lambda, n)(\eta + 2\epsilon)$-isotopy, where $f(\lambda, n)$ is some nonnegative function of λ and n; in other words, the diameter of the path traversed by each point of gI_2^n under the isotopy X is less than $f(\lambda, n)(\eta + 2\epsilon)$.

5(X). If for some i', $1 \leq i' \leq n$, we have $g = e$ on $I_{i'}(d; r_0)$, where d is one of the points $d_{i',k}$ for $i' \neq i$ and one of the points d_1 and d_3 for $i' = i$, then $(X)_t = e$ on $I_{i'}(d; r_0)$ for all t.

The proof of 4(X) is easily carried out by induction on l. Indeed, by hypothesis, according to (*), we are given that the image of each parallelepiped $\Pi = \bigcap_{1 \leq i' \leq n} \Pi_{i'}(d', d''; r_0)$, where d' and d'' for a given $i' \neq i$ are equal to two adjacent values of $d_{i',k}$, and $d' = d_1$ and $d'' = d_3$ for $i' = i$, goes into the ϵ-neighborhood of this parallelepiped under g. If $l = \lambda$ then both σ and τ take each point of the parallelepiped into itself. At the same time, ρ is evidently a $(d_3 - d_1)$-isotopy. Thus each of the isotopies τ, σ, ρ is certainly an $n(\eta + 2\epsilon)$-isotopy, while ω_λ is a $4n(\eta + 2\epsilon)$-isotopy, and so we have verified that the induction can begin.

Thus we may assume that the image of Π under the homeomorphism $\tilde{g}_{l+1} = (\omega_{l+1})_1 g$ has diameter less than $f_{l+1}(n)(\eta + 2\epsilon)$, where f_{l+1} is some positive function of n. But again the isotopies τ_l and σ_l, which are constructed at the next step, map $\tilde{g}_{l+1}\Pi$ onto itself for each t, while ρ_l is a $(d_3 - d_1)$-isotopy. Therefore $\bar{\omega}_l$ is certainly a $4f_{l+1}(n)(\eta + 2\epsilon)$-isotopy, and so $\omega_\lambda \circ \cdots \circ \omega_l$ is an $\overline{f_l}(n)(\eta + 2\epsilon)$-isotopy, where $\overline{f_l}$ is a positive function of n depending on l.

Then X is an $f(\lambda, n)(\eta + 2\epsilon)$-isotopy, where $f(\lambda, n)$ is a function of λ and n, and 4(X) is proved.

The proof of 5(X) is almost obvious. For d_1 and d_3 the assertion follows from the fact that $X \equiv e$ on $gI_i(d; r_0)$ for $d = d_1$ and d_3, and for $i' \neq i$ from the fact that τ and σ take $gI_{i'}(d; r_0)$ into itself, while ρ takes $I_{i'}(d; r_0)$ into itself.

3.20. Remark. If we consider only homeomorphisms g which take $I_i(d_2; r_0)$ onto itself, then we can alter the constuction of X in such a way that for all t $(X)_t$ also takes $I_i(d_2; r_0)$ into itself. To do this it is obviously sufficient so to modify the construction of ω_λ alone, since the remaining isotopies ω_l, $l < \lambda$, are the identity on $I_i(d_2; r_0)$.

We modify the construction of ω_λ as follows: τ_λ is constructed as before, $\sigma_\lambda \equiv e$, and ρ is $\xi_i(d_1, c, c_\lambda + \epsilon, d_2; \bar{r} + \epsilon, r_0 - \epsilon)$, where c is chosen so that $I_i(c; r_0)$ lies between $(\tau)_1 gI_i(c_\lambda; r_0)$ and $I_i(d_2; r_0)$.

The verification of all the properties 1–5 (ω) is obvious here.

§4. Proof of the Local Theorem

4.1. We first show that the formulation given in 2.16 can be weakened, in a way which leads to *technical simplifications in the proof of the theorem.*

The first modification affects condition 3(L). We can replace it by the two weaker requirements

3(a). $\mathrm{H}(g) \equiv e$ on $R^n \setminus gI_2^n$,
3(b). $(\mathrm{H}(g))_t(\partial I^p \times \bar{I}_2^{n-p}) = \partial I^p \times \bar{I}_2^{n-p}$ for all t.

The main advantage from this change is that the reasoning effectively ceases to depend on p, and it is necessary only to ensure that condition 3(b) is satisfied, which is straightforward, as we shall see.

Using 3(a) and 3(b), we obtain an isotopy of the manifold $g(I^p \times \bar{I}_2^{n-p})$, and we can apply the method of 1.16 in order to replace it by an isotopy which is the identity on the boundary of this manifold.

To be precise: assume that for some $\delta > 0$ and for each δ-shift g: $I_2^n \to R^n$ which is the identity on $\partial I^p \times \bar{I}_2^{n-p}$ we have obtained an isotopy $\mathrm{H}(g)$ of the manifold $g(I^p \times \bar{I}_2^{n-p})$ such that

1) $\mathrm{H}(g)$ depends continuously on g;
2) $(\mathrm{H}(g))_0 = e$;
3) $\mathrm{H}(g) \equiv e$ on $g(I^p \times \partial\bar{I}_2^{n-p})$;
4) $(\mathrm{H}(g))_1 = g^{-1}$ on gI^n (in particular, $(\mathrm{H}(g))_1 = e$ on $\partial I^p \times \bar{I}^{n-p}$);
5) $(\mathrm{H}(g))_1 = e$, if $g = e$.

By induction on the dimension, we can assume that for some neighborhood of $e(\partial I^p \times \bar{I}_2^{n-p})$ on the group $\mathfrak{H}_c(\partial I^p \times \bar{I}_2^{n-p})$ a contraction $\widetilde{\mathrm{H}}(h)$ of this neighborhood into e is given, where by 1.16 and 1.11 we can assume that

1) if $h = e$ on $\partial I^p \times \partial\bar{I}_2^{n-p}$ then the whole isotopy

$$\widetilde{\mathrm{H}}(h) \equiv e \text{ on } \partial I^p \times \partial\bar{I}_2^{n-p};$$

2) if $h = e$ on $\partial I^p \times \bar{I}^{n-p}$ then the whole isotopy

$$\widetilde{\mathrm{H}}(h) \equiv e \text{ on } \partial I^p \times \bar{I}^{n-p};$$

3) if $h = e$ then $\widetilde{\mathrm{H}}(h) \equiv e$.

Then for a given g: $I_2^n \to R^n$ we have firstly the restriction $\mathrm{H}^\partial(g)$ of the isotopy $g^{-1}\mathrm{H}(g)g$ to $\partial I^p \times \bar{I}_2^{n-p}$, and for each t an isotopy $\widetilde{\mathrm{H}}((\mathrm{H}^\partial(g))_t)$. From this, as in 1.16, we obtain a "diagonal" family of isotopies $\mathrm{H}_s^\partial(g)$ on $\partial I^p \times \bar{I}_2^{n-p}$, with the properties:

1) it depends continuously on g;
2) $\mathrm{H}_0^\partial(g) = \mathrm{H}^\partial(g)$, $\mathrm{H}_1^\partial(g) = \mathrm{E}$;
3) $(\mathrm{H}_s^\partial(g))_0 = e$ for all g and s;
4) $\mathrm{H}_s^\partial(g) \equiv e$ on $\partial I^p \times \partial\bar{I}_2^{n-p}$ for all s;
5) $(\mathrm{H}_s^\partial(g))_1 = e$ on $\partial I^p \times \bar{I}^{n-p}$ for all s;

6) $(H_s^{\partial}(g))_1 = e$ if $g = e$.

Now let G: $((\partial I^p \times \overline{I}_2^{n-p}) \times [0, 1] \approx Q$ be the homeomorphism on the neighborhood $Q = Q(\partial I^p \times \overline{I}_2^{n-p})$ into $I^p \times \overline{I}_2^{n-p}$ determined by the conditions $G(x \times 0) = x$, $x \in \partial I^p \times \overline{I}_2^{n-p}$, and G maps $(x \times [0, 1])$ isometrically onto the segment of perpendicular from $x \in \partial I^p \times \overline{I}_2^{n-p}$ to $\overline{R}^{l-p}$. Let $Q_s = G((\partial I^p \times \overline{I}_2^{n-p}) \times s)$, and let P_s: $((\partial i^p \times \overline{I}_2^{n-p}) \times s) \approx \partial I^p \times \overline{I}_2^{n-p}$ be induced by the projection of the direct product.

We first specify a new isotopy $H'(g)$ of the manifold $g(I^p \times \overline{I}_2^{n-p})$ as follows:

$$H'(g) = H(g) \text{ on } g(I^p \times \overline{I}_2^{n-p}) \setminus gQ,$$
$$H'(g) = H(g) gGP_s^{-1} (H^{\partial}(g))^{-1} H_{1-s}^{\partial}(g) P_s G^{-1} g^{-1} \text{ on } Q_s.$$

It is easy to see that this does indeed define an isotopy with properties 1–6 (L). In fact, for $s = 0$ we obtain E, for $s = 1$ we obtain $H'(g) = H(g)$, and for $t = 1$ we obtain $(H'(g))_1 = (H(g))_1$, and in particular $(H'(g))_1 = g^{-1}$ on gI^n.

The second weakening of the formulation is given by the possibility of omitting 5 (L). To prove this we first take for each g the homeomorphism $u(g)$: $R^n \to R^n$ which is the identity on I^n, maps ∂I_2^n homothetically into the boundary of the maximal I_r^n lying in gI_2^n, and is linear on the complementary intervals on each ray from o. Then $u(g)$ depends continuously on g, and $u(e) = e$. It is clear that the isotopy $H'(g) = u\,(H(e))^{-1} u^{-1} H(g)$ possesses properties 1–5 if $H(g)$ possesses properties 1–4 of the theorem. Thus, finally, we shall prove our assertion in the following form:

Local Theorem. *There exists $\delta > 0$ such that for each homeomorphic δ-shift g: $I_2^n \to R^n$ there is an isotopy $H(g)$ such that*

1) $H(g)$ *depends continuously on* g;
2) $(H(g))_0 = e$;
3) $H(g) \equiv e$ *on* $R^n \setminus gI_2^n$;
4) $(H(g))_1 = g^{-1}$ *on* gI^n;
5) *if* $g = e$ *on* $\partial I^p \times \overline{I}_2^{n-p}$ *then for all* t
$$(H(g))_t(\partial I^p \times \overline{I}_2^{n-p}) = \partial I^p \times \overline{I}^{n-p}$$

It turns out that we may take $1/8 \cdot 3^{2n}$ *for* δ.

(We note that conditions 3 (L) a) and 3 (L) b) of the beginning of the present subsection coincide respectively with conditions 3) and 5) of this statement.)

4.2. We pass to the proof of the local theorem, which consists of the reduction of this theorem to the lemma on the correction of homeomorphisms proved in §3.

We construct a sequence of isotopies Φ_j depending continuously on g, for which the infinite composition will be defined (that is, $(\Phi_j)_0 = (\Phi_{j-1})_1$ for $j > 1$) and there is a limit isotopy, which will be taken as $H(g)$.

More precisely, we shall in fact construct isotopies $\overline{\Phi}_j(g)$, $j \geq 1$, for which $(\overline{\Phi}_j)_0 = e$ for all j, and we put $\Phi_1 = \overline{\Phi}_1$ and $\Phi_j = \overline{\Phi}_j (\Phi_{j-1})_1$. That the composition is defined is seen at once. We require the following conditions (given in form convenient for the inductive construction) to be satisfied:

1 (Φ). The isotopy Φ_j depends continuously on g;

2 (Φ). $(\Phi_j)_1 g\, I_i(d; r) \subset I_i^{\epsilon_j}(d; r+\epsilon_j)$, where $1 \le i \le n$, $r = 1 + 1/2^j$, $\epsilon_j = 1/2^{j+3} \cdot 3^{2n}$, and d runs through (a) all rational points in the interval $[-1 - 1/2^{j+1}, 1 + 1/2^{j+1}]$ which are multiples of $1/2^{j+1} \cdot 3^{n-1}$ and (b) the points dividing each of the intervals $[-1 - 1/2^j, -1 - 1/2^{j+1}]$ and $[1 + 1/2^{j+1}, 1 + 1/2^j]$ into $3^n - 1$ equal parts, including the end points;

3 (Φ). $\Phi_j \equiv (\Phi_{j-1})_1$ for $j > 1$ on $R^n \setminus (\Phi_{j-1})_1 g I^n_{1+1/2^{j-1}}$ and $\Phi_j \equiv e$ for $j = 1$ on $R^n \setminus g I^n_2$;

4 (Φ). Φ_j is a $\delta_{\Phi,j}$-isotopy, where $\delta_{\Phi,j} = \eta_\Phi / 2^j$ and η_Φ depends on n but not on g and j;

5 (Φ). If $g = e$ on $I_i(\pm 1; 2)$ for some i, $1 \le i \le n$, then for all t we have $(\Phi_j(g))_t I_i(\pm 1; 2) = I_i(\pm 1; 2)$.

4.3. Let us show that if isotopies Φ_j with these properties have been constructed then their infinite composition can be completed to a limit isotopy, which can be taken as H.

From 4(Φ) it follows that the limit mapping is defined, and thus Φ_j possesses a limit pseudo-isotopy, which we henceforth denote by $\mathrm{H}(g)$. It must be shown that $(\mathrm{H}(g))_1$ is a homeomorphism and that $\mathrm{H}(g)$ satisfies requirements 1)–5) of the Local Theorem as formulated in 4.2.

We show that $(\mathrm{H}(g))_1$ is a homeomorphism. By 3(Φ), Φ_1 is the identity on $R^n \setminus g I^n_2$, and therefore so are all the subsequent isotopies. Therefore $(\mathrm{H}(g))_1$ is a homeomorphism outside $g I^n_2$. Further, condition 3) of the theorem is satisfied. Since, by 3(Φ), $\Phi_j \equiv (\Phi_{j_0})_1$ for $j \ge j_0$ on $R^n \setminus (\Phi_{j_0})_1 g I^n_{1+1/2^{j_0}}$, we have $(\mathrm{H}(g))_1 = (\Phi_{j_0})_1$ on $R^n \setminus g I^n_{1+1/2^{j_0}}$, and so $(\mathrm{H}(g))_1$ is a homeomorphism on $R^n \setminus g I^n = \bigcup_{j=1}^\infty (R^n \setminus g I^n_{1+1/2^j})$.

It follows from 2(Φ), applied to the extreme values $d = \pm(1 + 1/2^{j_0})$ for all i, $1 \le i \le n$, that

$$I^n_{1+\frac{1}{2^{j_0}}+\varepsilon_{j_0}} \supset (\Phi_{j_0})_1 g I^n_{1+\frac{1}{2^{j_0}}} \supset I^n_{1+\frac{1}{2^{j_0}}-\varepsilon},$$

and since

$$\bigcap_{j=1}^{\infty} I^n_{1+\frac{1}{2^j}+\varepsilon_j} = \bigcap_{j=1}^{\infty} I^n_{1+\frac{1}{2^j}-\varepsilon_j} = I^n,$$

we have

$$(\mathrm{H}(g))_1 g (R^n \setminus I^n) = R^n \setminus I^n. \tag{1}$$

If the point $x \in I^n$, then we take a sufficiently large j and consider the minimal cube $I^n(x)$ containing the point x in its interior and each side of which is orthogonal to one of the axes at a point with coordinate a multiple of $1/2^{j+1}$.

We observe that it follows from 2(Φ) that the image of the slab $\Pi_i(d_1, d_2; r_0)$, where $1 \le i \le n$, $r_0 = 1 + 1/2^{j+1}$, and d_1, d_2 take the values referred to in 2(Φ), lies in $\Pi_i(d_1 - \epsilon_j, d_2 + \epsilon_j; r_0 + \epsilon_j)$ (compare the proof of relation (5) in 3.5). Hence if we take an arbitrary parallelepiped $\bigcap_{1\le i\le n}\Pi_i(d_{i,1}, d_{i,2}; r_0)$, where $d_{i,1}$ and $d_{i,2}$ run through the numbers referred to in $2(\Phi)$ (a), then its image lies in its $(2\epsilon_j)$-neighborhood.

By the above, $(\Phi_j)_1 g$ takes the cube $I^n(x)$ into its $(2\epsilon_j)$-neighborhood, while the diameter of the cube is not greater than $n/2^j$ and none of the subsequent isotopies take its image out of the $\eta_\Phi/2^{j-2}$-neighborhood of $(\Phi_j)_1 gI^n(x)$ (by $4(\Phi)$). Hence $x = \lim(\Phi_j)_1 gx$, and so $(H(g))_1 x = g^{-1}x$.

Thus, first, $(H(g))_1$ is g^{-1} on gI^n; that is, condition (4) of the theorem is satisfied, and second, $(H(g))_1$ is a homeomorphism on gI^n, and, in view of (1), everywhere.

Condition 2) of the theorem is obvious, while 5) follows from $5(\Phi)$.

It remains to verify that $H(g)$ depends continuously on g. It must be shown that for every homeomorphic δ-shift of I_2^n in R^n and for every $\epsilon > 0$ there is a neighborhood $\Omega(g)$ in the space of homeomorphic mappings of I_2^n in R^n, such that if $g' \in \Omega(g)$ then for all $x \in R^n$ and $t \in [0, 1]$

$$\rho((H(g))_t x, (H(g'))_t x) < \varepsilon.$$

Since $\Phi_j = E$ outside a compact subset of R^n, say outside $I^n_{2+\delta}$, for all $j \ge 1$, it is sufficient to show that for every g and for every point $(x, t) \in R^n - [0, 1]$ there is a neighborhood $\Omega_{x,t}(g)$ and a neighborhood $O_g(x, t)$ such that if $g' \in \Omega_{x,t}(g)$ and $(x', t') \in O_g(x, t)$ then $\rho((H(g'))_{t'} x', (H(g))_{t'} x') < \epsilon$. (Indeed, for a given g we can then construct a finite number of neighborhoods $O_g(x, t)$, say $O_{g,1}$, $O_{g,2}, \cdots, O_{g,l}$, covering $I^n_{2+\delta} \times [0, 1]$, and as $\Omega(g)$ we can take the intersection of the corresponding neighborhoods $\Omega_1, \Omega_2, \cdots, \Omega_l$.)

First let the point $x \in I^n_{2+\delta} \setminus gI^n$. Then there are $j_0 \ge 1$ and neighborhoods $O'_g(x)$ and $\Omega'_x(g)$ such that $O'_g(x) \subset R^n \setminus g' I^n_{1+2^{j_0}}$ for any $g' \in \Omega'_x(g)$. By $3(\Phi)$, all the isotopies $\Phi_j(g') \equiv (\Phi_{j_0}(g'))_1$ on $(\Phi_{j_0}(g'))_1 O'_g(x)$ for $j > j_0$. Thus $H(g')$ is determined on $O'(x) \times [0, 1]$ by the first j_0 isotopies $\overline{\Phi}_j(g')$ alone (that is, for some t_0, $0 \le t_0 < 1$, $(H(g))_t x' = (H(g'))_{t_0} x'$ for $t \ge t_0$, $g' \in \Omega'_x(g)$, and $x' \in O'_g(x)$). But by $1(\Phi)$ all the Φ_j depend continuously on g, so we can find neighborhoods $O''_g(x) \subset O'_g(x)$ and $\Omega''_x(g) \subset \Omega'_x(g)$ such that $\rho((H(g'))_t x', (H(g))_t x') < \epsilon$ for $t \in [0, 1]$, $g' \in \Omega''_x(g)$, $x' \in O''_g(x)$.

We see that as $\Omega_g(x, t)$ for any t we can take $O''_g(x) \times [0, 1]$, and as $\Omega_{x,t}(g)$ we can take $\Omega''_x(g)$.

If $t < 1$ then $H(g)$ is defined in the neighborhood of the point (x, t) by one or two isotopies $\overline{\Phi}_j$, and again the continuity of $H(g)$ follows here from $1(\Phi)$.

Finally, let $x \in gI^n$ and $t = 1$. First of all, by condition $4(\Phi)$ we find a j_0 such that for each $j \ge j_0$ and all $t \in [0, 1]$ we have, independently of g,

$$\rho((\Phi_j)_t x', (\Phi_{j_0})_1 x') < \frac{\varepsilon}{3}, \quad x' \in R^n.$$

If t_0 corresponds to the homeomorphism $(\Phi_{j_0})_1$ in $H(g)$ (that is, $(H(g))_{t_0} = (\Phi_{j_0})_1$), then this means that for $t > t_0$

$$\rho((H(g))_t x', (H(g))_{t_0} x') < \frac{\varepsilon}{3} \tag{2}$$

for all $x' \in R^n$ and independently of g.

Since $(\Phi_{j_0}(g))_1$ depends continuously on g, we can find neighborhoods $O'_g(x)$ and $\Omega'_x(g)$ such that $\rho((\Phi_{j_0}(g'))_1, (\Phi_{j_0}(g))_1 x') < \epsilon/3$ for $x' \in O'_g(x)$ and $g' \in \Omega'_x(g)$, and then

$$\rho((\mathrm{H}(g'))_{t_0} x',\ (\mathrm{H}(g))_{t_0} x') < \frac{\varepsilon}{3}. \tag{3}$$

From (2) and (3) we obtain for $t' \geq t_0$

$$\rho((\mathrm{H}(g'))_{t'} x',\ (\mathrm{H}(g))_{t'} x') < \rho((\mathrm{H}(g'))_{t_0} x', (\mathrm{H}(g))_{t_0} x') + \frac{2\varepsilon}{3} < \varepsilon.$$

We put $O_g(x, 1) = O'_g(x) \times [0, 1]$ and $\Omega_{x,1}(g) = \Omega'_x(g)$.

4.4. We pass to the construction of the isotopies $\overline{\Phi}_j$. *Arguing by induction, suppose that the con*struction of the first $j-1$ steps has already been made, and so we have a homeomorphic shift g_{j-1}: $I^n_2 \to R^n$, equal to g if $j = 1$ and equal to $(\Phi_{j-1})_1 g$ if $j > 1$, for which the following conditions are evidently satisfied:

1 (g). g_{j-1} depends continuously on g;

2 (g). $g_{j-1} I_i(d; r) \subset I_i^{\varepsilon_{j-1}}(d; r + \varepsilon_{j-1})$, where $1 \leqslant i \leqslant n$, $\quad r = 1 + \frac{1}{2^{j-1}}$,

$\epsilon_{j-1} = 1/2^{j+1} \cdot 3^{2n}$, and d runs through the values referred to in $2(\Phi)$, with j replaced by $j-1$;

$3(\tilde{g})$. if for some i, $1 \leq i \leq n$, we have $g = e$ on $I_i(\pm 1; 2)$, then $g_{j-1} I_i(\pm 1; 2) = I_i(\pm 1; 2)$.

We shall construct $\overline{\Phi}_j$ on the basis of these properties of g_{j-1} alone. We observe that for $j = 1$ every homeomorphic δ-shift g, where $\delta = 1/8 \cdot 3^{2n}$, possesses these properties.

4.5. We shall construct $\overline{\Phi}_j$ as the product of two isotopies: $\overline{\Phi}_j = \psi_j \phi_j$, where ϕ_j in its turn is a composition of isotopies: $\phi_{j,1} \circ \cdots \circ \phi_{j,n}$, one for each axis. The purpose of the isotopy ϕ_j is roughly speaking to double the number of cubes $I_i(d; r)$ which are correct in the sense that their *images lie in the slabs* $I_i^{\epsilon_{j-1}}(d; r + \epsilon_{j-1})$ *of thickness* $2\epsilon_{j-1}$, and the purpose of ψ_j *is to decrease the* width of these slabs from $2\epsilon_{j-1}$ to $2\epsilon_j$. The purpose of $\phi_{j,i}$ is to multiply the number of "correct" cubes orthogonal to the ith axis, but, as explained earlier, not just by two but with something in reserve, in fact roughly by $2 \cdot 3^{n-1}$, since at the same time as the number of "correct" cubes is being increased for the i-axis, about two thirds of the cubes are being lost for the other directions.

4.6. The isotopy ψ_j can be constructed at once: it is equal to the product of the isotopies $\zeta_{j,i,d}$, taken in an arbitrary order, where (see 3.3)

$$\zeta_{j,i,d} = v_i\left(d;\ \varepsilon_{j-1},\ \varepsilon_j,\ 2\varepsilon_{j-1};\ 1 + \frac{1}{2^j} + \varepsilon_{j-1},\ 1 + \frac{1}{2^{j-1}} - \varepsilon_{j-1}\right),$$

and d runs through the numbers referred to in $2(\Phi)$.

4.7. **Remark.** The isotopies $\zeta_{j,i,d}$ and $\zeta_{j,i,d'}$ have disjoint supports if $d \neq d'$, since the distance between adjacent numbers d in condition $2(\Phi)$ is $1/2^j \cdot 3^{n-1}$ or $1/2^j(3^n - 1)$, which is not less than $4\epsilon_{j-1} = 1/2^j \cdot 3^{2n}$.

4.8. The following properties of the $\zeta_{j,i,d}$ follow directly from the definition:

1 (ζ). $\zeta_{j,i,d}$ is independent of g;

2 (ζ). $(\zeta_{j,i,d})_1\, I_i^{\varepsilon_{j-1}}\left(d;\ 1+\frac{1}{2^j}+\varepsilon_{j-1}\right) = I_i^{\varepsilon_j}\left(d;\ 1+\frac{1}{2^j}+\varepsilon_{j-1}\right)$;

3 (ζ). $\zeta_{j,i,d} \equiv e$ on $R^n \diagdown I^n_{1+\frac{1}{2^{j-1}}-\varepsilon_{j-1}}$;

4 (ζ). $\zeta_{j,i,d}$ is an $(\varepsilon_{j-1}-\varepsilon_j)$-isotopy;

5 (ζ). $(\zeta_{j,i,d})_t$ maps $I_{i'}(\pm 1;\ 2)$ onto itself for all t and i.

From these properties the following properties of the isotopy ψ_j follow easily:

1 (ψ). ψ_j is independent of g;

2 (ψ). $(\psi_j)_1\, I_i^{\varepsilon_{j-1}}\left(d;\ 1+\frac{1}{2^j}+\varepsilon_{j-1}\right) \subset I_i^{\varepsilon_j}\left(d;\ 1+\frac{1}{2^j}+\varepsilon_j\right)$;

3 (ψ). $\psi_j \equiv e$ on $R^n \diagdown I^n_{1+\frac{1}{2^{j-1}}-\varepsilon_{j-1}}$;

4 (ψ). ψ_j is an $n(\varepsilon_{j-1}-\varepsilon_j)$-isotopy;

5 (ψ). $(\psi_j)_t$ maps $I_i(\pm 1;\ 2)$ onto itself for all t and i.

Properties $1(\psi)$ and 3–5 (ψ) follow at once from the corresponding properties (ζ) and the definition of ψ_j. Let us prove $2(\psi)$ for a given pair $(i,\ d)$.

We observe that for $i' \neq i$ the isotopy $\zeta_{j,i',d}$ takes each of $I_i^{\epsilon_{j-1}}(d;\ 1+1/2^j+\epsilon_{j-1})$ and $I_i^{\epsilon_j}(d;\ 1+1/2^j+\epsilon_{j-1})$ into itself, that for $i'=i$ and $d' \neq d$ it is the identity on these sets, and finally that

$$\zeta_{j,i,d}\, I_i^{\varepsilon_{j-1}}\left(d;\ 1+\frac{1}{2^j}+\varepsilon_{j-2}\right) = I_i^{\varepsilon_{j-1}}\left(d;\ 1+\frac{1}{2^j}+\varepsilon_{j-1}\right).$$

Hence

$$(\psi_j)_1\, I_i^{\varepsilon_{j-1}}\left(d;\ 1+\frac{1}{2^j}+\varepsilon_{j-1}\right) \subset I_i^{\varepsilon_j}\left(d;\ 1+\frac{1}{2^j}+\varepsilon_{j-1}\right). \tag{4}$$

Further, if $i' \neq i$ then for each $i'' \neq i'$ the isotopy $\zeta_{j,i'',d}$ takes

$$\Pi_{i'}\left(-1-\frac{1}{2^j}-\varepsilon_{j-1},\ 1+\frac{1}{2^j}+\varepsilon_j;\ 1+\frac{1}{2^j}+\varepsilon_{j-1}\right),$$

$$\Pi_{i'}\left(-1-\frac{1}{2^j}-\varepsilon_j,\quad 1+\frac{1}{2^j}+\varepsilon_{j-1};\ 1+\frac{1}{2^j}+\varepsilon_{j-1}\right) \text{and } I^n_{1+\frac{1}{2^j}+\varepsilon_{j-1}}$$

into themselves, and the same is true for $\zeta_{j,i',d}$ if $d \neq \pm(1+1/2^j)$. If $d = 1+1/2^j$, say, then

$$\zeta_{j,i',d}\, I^n_{1+\frac{1}{2^j}+\varepsilon_{j-1}} = \Pi_{i'}\left(-1-\frac{1}{2^j}-\varepsilon_{j-1},\quad 1+\frac{1}{2^j}+\varepsilon_j;\quad 1+\frac{1}{2^j}+\varepsilon_{j-1}\right),$$

and similarly for $d = -1-1/2^j$. Hence

$$(\psi_j)_1 I^n_{1+\frac{1}{2^j}+\varepsilon_{j-1}} \subset \bigcap_{i'\neq i} \Pi_{i'}\left(-1-\frac{1}{2^j}-\varepsilon_j;\ 1+\frac{1}{2^j}+\varepsilon_j;\ 1+\frac{1}{2^j}+\varepsilon_{j-1}\right)$$

and then

$$(\psi_j)_1 I_i^{\varepsilon_{j-1}}\left(d;\ 1+\frac{1}{2^j}+\varepsilon_{j-1}\right) \subset \bigcap_{i'\neq i} R_{i'}\left(-1-\frac{1}{2^j}-\varepsilon_j;\ 1+\frac{1}{2^j}+\varepsilon_j\right). \tag{5}$$

From the inclusions (4) and (5) it follows that

$$(\psi_j)_1 I_i^{\varepsilon_{j-1}}\left(d;\ 1+\frac{1}{2^j}+\varepsilon_{j-1}\right) \subset I_i^{\varepsilon_j}\left(d;\ 1+\frac{1}{2^j}+\varepsilon_j\right),$$

and it is clear that in fact there is equality.

4.9. We pass to the construction of the isotopies $\phi_{j,i}$, $1 \le i \le n$. We shall construct them successively in such a way that the following conditions are satisfied (here and later we denote by r_i the number $1 + 1/2^{j-1} - (3^i - 1)/2^j(3^n - 1)$, $0 \le i \le n$):

1 (ϕ). $\phi_{j,i}$ depends continuously on g;

2 (ϕ). $(\phi_{j,i})_1 g_{j-1} I_{i'}(d;\ r_i) \subset I_{i'}^{\epsilon_{j-1}}(d;\ r_i + \epsilon_{j-1})$, where $1 \le i \le n$, $1 \le i' \le n$, and d runs through

(1) for all i': the points dividing each of the intervals $[-r_i, -r_n]$, $[r_n, r_i]$ into $3^{n-i} - 1$ equal parts, including the end points, and also

(2) for $1 \le i' \le i$: a) the multiples of $1/2^{j+2} \cdot 3^{2n-i-1}$ in the interval

$$\left[-1-\frac{1}{2^{j+1}},\ 1+\frac{1}{2^{j+1}}\right],$$

b) the points dividing each of the intervals

$$\left[-r_n,\ -1-\frac{1}{2^{j+1}}\right] \text{and} \left[1+\frac{1}{2^{j+1}},\ r_n\right],$$

into $3^{n-i}(3^n - 1)$ *equal parts, including the end points;*

(3) for $i + 1 \le i' \le n$: the multiples of $1/2^j \cdot 3^{n-i-1}$ in the interval $[-r_n, r_n]$;

3 (ϕ). $\phi_{j,i} \equiv e$ on $R^n \setminus (\phi_{j,i-1})_1 g_{j-1} I^n_{r_{i-1}}$ for $i > 1$, and on $R^n \setminus g_{j-1} I^n_{r_0}$ for $i = 1$;

4 (ϕ). $\phi_{j,i}$ is a $\delta_{j,i}$-isotopy, where $\delta_{j,i} = \eta_i/2^j$ and η_i depends on i but not on j or g;

5 (ϕ). if $g_{j-1} I_{i'}(\pm 1;\ 2) = I_{i'}(\pm 1;\ 2)$ for some i', $1 \le i' \le n$, then for all $t \in [0, 1]$ we have $(\phi_{j,i})_t I_{i'}(\pm 1;\ 2) = I_{i'}(\pm 1;\ 2)$.

4.10. Let us show that the $\Phi_j = \overline{\Phi}_j(\Phi_{j-1})_1 = \psi_j \phi_j (\Phi_{j-1})_1$ satisfy the conditions 1–5 (Φ) if the isotopies $\phi_{j,i}$ possess properties 1–5 (ϕ) and the ψ_j properties 1–5 (ψ).

Clearly 1 (Φ), 4 (Φ), and 5 (Φ) follow from the corresponding properties of the ψ_j and ϕ_j. We show that 2(Φ) follows from 2(ψ) and 2(ϕ) for $i = n$. In fact, by the latter,

$$(\varphi_{j,n})_1 g_{j-1} I_i(d;\ r_n) \subset I_i^{\varepsilon_{j-1}}(d;\ r_n + \varepsilon_{j-1}),$$

where $1 \leq i \leq n$, $r_n = 1 + 1/2^j$, and d runs through exactly the values referred to in 2 (Φ) (the numbers in conditions (1) and (3) of 2 (ϕ) disappear, while those in conditions (2)a) and (2)b) become the numbers in conditions 2 (Φ)a) and 2 (Φ)b) respectively).

By 2(ψ), we have

$$(\Phi_j)_1 \, g_{j-1} I_i \left(d;\, 1 + \frac{1}{2^j}\right) = (\psi_j)_1 \, (\varphi_{j,n})_1 \, (\Phi_{j-1})_1 \, g_{j-1} \, I_i \left(d;\, 1 + \frac{1}{2^j}\right) \subset I_i^{\varepsilon_{j-1}}(d;\, r_n + \varepsilon_{j-1}).$$

Property 3 (Φ) follows from 3 (ϕ), 3 (ψ), and the fact that

$$R^n \setminus I^n_{1+\frac{1}{2^{j-1}}-\varepsilon_{j-1}} \supset R^n \setminus g_{j-1} I^n_{1+\frac{1}{2^{j-1}}}$$

by 2($\tilde{g}$).

4.11. We introduce some notation. Let r_i, $0 \leq i \leq n$, as above, denote the number

$$1 + \frac{1}{2^{j-1}} - \frac{3^i - 1}{2^j (3^n - 1)},$$

in particular,

$$r_0 = 1 + \frac{1}{2^{j-1}}, \quad r_n = 1 + \frac{1}{2^j}.$$

We shall call the numbers d in condition 2 (ϕ), related to i', points of division of order (j, i, i'), where, since j is fixed, we may omit the index j and write (i, i'). Thus for $i' > i$ the numbers of order (i, i') are

a) the multiples of $1/2^i \cdot 3^{n-1}$ in $[-r_n, r_n]$,

b) the numbers dividing each of the intervals $[-r_i, -r_n]$ and $[r_n, r_1]$ into $3^{n-1} - 1$ equal parts;

and for $i' \leq i$

a) the multiples of $1/2^{j+1} 3^{2n-i-1}$ in $[-r_n + 1/2^{j+1}, r_n - 1/2^{j+1}]$,

b) the numbers dividing each of the intervals $[-r_n, -r_n + 1/2^{j+1}]$, $[r_n - 1/2^{j+1}, r_n]$ into $3^{n-i}(3^n - 1)$ equal parts,

c) the numbers dividing each of the intervals $[-r_i, r_n]$, $[r_n, r_i]$ into $3^{n-i} - 1$ equal parts.

We note that the formally defined points of order $(j, 0, i')$ coincide with those of order $(j - 1, n, i')$, and also that in going from the points of order $(i - 1, i')$ to those of order (i, i'), $i \neq i'$, each third point remains.

Between r_{i-1} and r_i there is just one number of order $(i - 1, i')$ for all i', namely

$$\frac{r_{i-1} + r_i}{2} = 1 + \frac{1}{2^{j-1}} - \frac{2 \cdot 3^{i-1} - 1}{2^{j-1}(3^n - 1)}.$$

We denote it by $\bar{r}$.

4.12. Arguing by induction, suppose that the first $i - 1$ steps have been carried out, and so a homeomorphism $\bar{g}_{i-1}$ has been constructed, equal to g_{j-1} in the case $i=1$, and for $i>1$ equal to $(\phi_{j,i-1})_1 g_{j-1}$, where

1 $(\tilde{g})$. $\bar{g}_{i-1}$ depends continuously on g;

2 $(\bar{g})$. $\bar{g}_{i-1} I_{i'}(d; r_{i-1}) \subset I_{i'}^{\epsilon_j - 1}(d; r_{i-1} + \epsilon_{j-1})$, where $1 \le i' \le n$ and d runs through the points of order $(i-1, i')$;

3 $(\bar{g})$. If for some i', $1 \le i' \le n$, $g_{j-1} I_{i'}(\pm 1; 2) = I_{i'}(\pm 1; 2)$, then $\bar{g}_{i-1} I_{i'}(\pm 1; 2) = I_{i'}(\pm 1; 2)$.

These properties follow for $i=1$ from conditions 1–3 (g_{j-1}), and for $i>1$ from the induction hypotheses on $\phi_{j,i-1}$, namely 1 (ϕ), 2 (ϕ), and 5 (ϕ), with i replaced by $i-1$.

We see that the induction can be started.

4.13. We mention again that the purpose of the isotopy $\phi_{j,i}$ is to multiply by approximately $2 \cdot 3^{n-1}$ the number of "correct" cubes of the form $I_{i'}(d; r_i)$. We shall base this construction on the lemma of §3. In this construction, as we know, there is a violation of the conditions $2(\bar{g})$ for "correct" cubes in the orthogonal directions. Namely, the image of $I_{i'}(d; r_i)$, $i' \neq i$, goes outside the zone $I_{i'}^{\epsilon_j - 1}(d; r_i + \epsilon_{j-1})$, although it remains in $\Pi_{i'}(d' - \epsilon_{j-1}, d'' + \epsilon_{j-1}; r_i + \epsilon_{j-1})$, where d' and d'' are the points of order $(i-1, i')$ adjacent to d. In the construction of $\psi_{j,i}$ we take these violations into account and make the necessary corrections. However we cannot correct the situation for all the cubes $I_{i'}(d; r_i)$, where $d \in [-r_i, r_i]$ has order $(i-1, i')$, since the images of these cubes cease to be separated by hyperplanes of the form $R_{i',a}$. However the images of the cubes $I_{i'}(d; r_i)$, where d runs through the points of order (i, i'), are nevertheless separated, and we can arrange that for them the condition of type $2(\bar{g})$ is again satisfied.

We construct $\phi_{j,i}$ as the product of an isotopy $\bar{\phi}_{j,i}$ for which $(\bar{\phi}_{j,i})_0 = e$, and a homeomorphism equal to $(\bar{\phi}_{j,i-1})_1$ for $i>1$ and equal to e for $i=1$. In its turn, in accordance with what has been said above, $\bar{\phi}_{j,i}$ is constructed as the product of two isotopies, $\bar{\phi}_{j,i} = \tilde{\psi}_{j,i} \tilde{\phi}_{j,i}$, the purpose of $\tilde{\phi}_{j,i}$ being to multiply by roughly $2 \cdot 3^{n-1}$ the number of "correct" cubes orthogonal to the ith axis, and of $\tilde{\psi}_{j,i}$ being to correct the violations which arise in the orthogonal directions, this correction being made only for the points of order (i, i') for all $i' \neq i$, so that we lose approximately two thirds of the correct cubes in each of these directions.

It is clear that as a result of the construction of all the isotopies $\phi_{j,i}$, $1 \le i \le n$, for each axis there is one multiplication by $2 \cdot 3^{n-1}$ of the number of correct cubes and $n-1$ times a reduction to one third. Thus altogether the number of correct cubes is (roughly) doubled for each axis.

4.14. We construct the correcting isotopy $\tilde{\psi}_{j,i}$ at once. We form it as the product of the isotopies $\tilde{\zeta}_{j,i',d}$, taken in an arbitrary order, where

$$\zeta_{j,i',d} = \xi_{i'}(d' - 2\varepsilon_{j-1},\ d' - \varepsilon_{j-1},\ d - \varepsilon_{j-1},\ d;\ \bar{r} + \varepsilon_{j-1},\ r_{i-1} - \varepsilon_{j-1})$$

$$\times\ \xi_{i'}(d,\ d'' + \varepsilon_{j-1},\ d + \varepsilon_{j-1},\ d'' + 2\varepsilon_{j-1};\ \bar{r} + \varepsilon_{j-1},\ r_{i-1} - \varepsilon_{j-1}),$$

$1 \le i' \le n$, $i' \neq i$, d runs through the points of order (i, i'), and d', d'' are the points of order $(i-1, i')$ adjacent to d on the left and right respectively.

Since the distance between neighboring points of order $(i-1, i')$ can only be

$$\frac{3^{i-1}}{2^j \cdot (3^i - 1)}, \quad \frac{1}{2^{j+1} \cdot 3^{2i-i}}, \quad \frac{1}{2^{j+1} \cdot 3^{i-i-1} \cdot (3^n - 1)}, \text{ or } \frac{1}{2^j \cdot 3^{i-i}}$$

and none of these numbers is greater than $1/2^j$, it follows that each $\tilde{\zeta}_{j,i',d}$ is a $1/2^j$-isotopy. Hence $\tilde{\psi}_{j,i}$ is an $n/2^j$-isotopy. Moreover, none of these numbers is less than $4\epsilon_{j-1} = 1/2^{j+1} \cdot 3^{2n}$, and therefore the supports of the isotopies $\tilde{\zeta}_{j,i',d}$ and $\tilde{\zeta}_{j,i',d'}$ are disjoint.

4.15. The $\tilde{\psi}_{j,i}$ possess the following properties, which, like those of ψ_j, follow easily from the properties of isotopies of type ξ and the definition of the $\tilde{\psi}_{j,i}$:

1 $(\tilde{\psi})$. $\tilde{\psi}_j$ is independent of g;

2 $(\tilde{\psi})$ (1). $(\tilde{\psi}_{j,i})_1 I_i^{\epsilon_{j-1}}(d; \bar{r} + \epsilon_{j-1}) = I_i^{\epsilon_{j-1}}(d, r_i + \epsilon_{j-1})$ or all numbers d of order (i, i),

2 $(\tilde{\psi})$ (2). $(\tilde{\psi}_{j,i})_1(\Pi_{i'}(d' - \epsilon_{j-1}, d'' + \epsilon_{j-1}; \bar{r} + \epsilon_{j-1}) \cap R_i(-r_i - \epsilon_{j-1}, r_i + \epsilon_{j-1})) \subset I_{i'}^{\epsilon_{j-1}}(d; r_i + \epsilon_{j-1})$, where d, d', d'', i' are as in the construction of $\tilde{\psi}_{j,i}$ in 4.14;

3 $(\tilde{\psi})$. $\tilde{\psi}_{j,i} \equiv e$ on $R^n \setminus I^n_{r_{i-1}-\varepsilon_{j-1}}$;

4 $(\tilde{\psi})$. $\tilde{\psi}_{j,i}$ is an $\dfrac{n}{2^j}$-isotopy;

5 $(\tilde{\psi})$. $(\tilde{\psi}_{j,i})_t I_{i'}(\pm 1, 2) = I_{i'}(\pm 1; 2)$ for all t and i'.

4.16. The isotopy $\tilde{\phi}_{j,i}$ is constructed so as to fulfill the following requirements:

1 $(\tilde{\phi})$. $\tilde{\phi}_{j,i}$ depends continuously on g;

2 $(\tilde{\phi})$ (1). $(\tilde{\phi}_{j,i})_1 \bar{g}_{i-1} I_i(d; r_i) \subset I_i^{\epsilon_{j-1}}(d; \bar{r} + \epsilon_{j-1})$, where d runs through the points of order (i, i),

2 $(\tilde{\phi})$ (2). $(\tilde{\phi}_{j,i})_1 \bar{g}_{i-1} I_{i'}(d; r_i) \subset \Pi_{i'}(d' - \epsilon_{j-1}, d'' + \epsilon_{j-1}; \bar{r} + \epsilon_{j-1}) \cap R_i(-r_i - \epsilon_{j-1}, r_i + \epsilon_{j-1})$, where d runs through the points of order (i, i'), $i' \neq i$, and d', d'' are the points of order $(i-1, i')$ adjacent to d;

3 $(\tilde{\phi})$. $\tilde{\phi}_{j,i} \equiv e$ on $R^n \setminus \bar{g}_{i-1} I^n_{r_{i-1}}$;

4 $(\tilde{\phi})$. $\tilde{\phi}_{j,i}$ is a $\tilde{\delta}_{j,i}$-isotopy, where $\tilde{\delta}_{j,i} = \tilde{\eta}_i / 2^j$ and $\tilde{\eta}_i$ is independent of j and g;

5 $(\tilde{\phi})$. if $\bar{g}_{i-1} I_{i'}(\pm 1; 2) = I_{i'}(\pm 1; 2)$ for some i', $1 \leq i' \leq n$, then for all t we have $(\tilde{\phi}_{j,i})_t I_{i'}(\pm 1; 2) = I_{i'}(\pm 1; 2)$.

4.17. Let us show that conditions (ϕ) follow from the conditions $(\tilde{\phi})$ and $(\tilde{\psi})$. We recall that $\phi_{j,i} = \tilde{\psi}_{j,i}(\phi_{j,i-1})_1$ for $i > 1$ and $\phi_{j,1} = \tilde{\psi}_{j,i}\tilde{\phi}_{j,i}$ for $i = 1$. The conditions 1, 2, 4, 5 (ϕ) follow directly from the corresponding conditions $(\tilde{\phi})$ and $(\tilde{\psi})$. Condition 3 (ϕ) follows from 3 $(\tilde{\phi})$ and 3 $(\tilde{\psi})$, since by 2 $(\bar{g})$ it is obvious that $\bar{g}_{i-1}(\partial I^n_{r_{i-1}}) \subset R^n \setminus I^n_{r_{i-1}-\epsilon_{j-1}}$, and so $R^n \setminus \bar{g}_{i-1} I^n_{r_{i-1}} \subset R^n \setminus I^n_{r_{i-1}-\epsilon_{j-1}}$. But $\tilde{\phi}_{j,i} \equiv e$ on $R^n \setminus \bar{g}_{i-1} I^n_{r_{i-1}}$, while $\psi_{j,i} \equiv e$ on $R^n \setminus I^n_{r_{i-1}-\epsilon_{j-1}}$.

4.18. We construct $\tilde{\phi}_{j,i}$ as the composition of κ isotopies $\chi_{j,i,k}$, $1 \leq k \leq \kappa$, where $\kappa + 1$ is the number of points of order $(i-1, i')$ in the interval $[-r_n, r_n]$, including the end points $(\kappa = 2 \cdot 3^{n-1}(2^j + 1) - 2)$.

In its turn, $\chi_{j,i,k}$ is constructed as the product of an isotopy $\bar{\chi}_{j,i,k}$, where $(\bar{\chi}_{j,i,k})_0 = e$, and a homeomorphism equal to e for $k=1$ and to $(\chi_{j,i,k-1})_1$ for $k>1$.

Let $d_0, d_1, \cdots, d_\kappa$ be the division points of order $(i-1, i)$ in $[-r_n, r_n]$. One isotopy $\overline{\chi}_{j,i,k}$ is constructed for each interval $[d_{k-1}, d_k]$, $1 \le k \le \kappa$, the interval $[d_k, d_{k+1}]$ also being employed in the construction of $\overline{\chi}_{j,i,k}$. For the case $k=\kappa$ we take as $d_{\kappa+1}$ the neighboring point of order $(i-1, i)$ on the right of $d_\kappa = r_n$, and as d_{-1} we take the neighboring point of order $(i-1, i)$ on the left of $d_0 = -r_n$. We construct $\overline{\chi}_{j,i,k}$ so as to fulfill the requirements

1 (χ). *$\chi_{j,i,k}$ depends continuously on g;*

2 (χ) (1). $(\chi_{j,i,k})_1 \overline{g}_{i-1} I_i(d; r_i) \subset I_i^{\epsilon_{j-1}}(d; \overline{r}+\epsilon_{j-1})$, where d runs through the points of order (i, i) in $[d_{k-1}, d_k]$;

2 (χ) (2). $(\chi_{j,i,k})_1 \overline{g}_{i-1} I_{i'}(d; r_i) \subset \Pi_{i'}(d'-\epsilon_{j-1}, d''+\epsilon_{j-1}; \overline{r}+\epsilon_{j-1}) \cap R_i(-r_i-\epsilon_{j-1}, r_i+\epsilon_{j-1})$, where $1 \le i' \le n$, $i' \ne i$, and d', d'' are the points of order $(i-1, i')$ adjacent to d;

3 (χ). $\overline{\chi}_{j,i,k} \equiv e$ on $R^n \backslash \overline{g}_{i-1}\Pi_i(d_{k-1}, d_{k+1}; r_{i-1})$, $(\overline{\chi}_{j,i,k})_1 = e$ on $R^n \backslash \overline{g}_{i-1}\Pi_i(d_{k-1}, d_k; r_{i-1})$;

4 (χ). $\overline{\chi}_{j,i,k}$ is a $\delta_{j,i,k}$-isotopy, where $\delta_{j,i,k} = \eta_{i,k}/2^j$ and $\eta_{i,k}$ is independent of j, k, and g;

5 (χ). if $\overline{g}_{i-1} I_{i'}(\pm 1; 2) = I_{i'}(\pm 1; 2)$ for some i', $1 \le i' \le n$, then $(\overline{\chi}_{j,i,k})_t I_{i'}(\pm 1; 2) = I_i(\pm 1; 2)$ for all t and all k.

4.19. Conditions $(\widetilde{\phi})$ follow from conditions (χ), taken for all k, $1 \le k \le \kappa$, if we define $\widetilde{\phi}_{j,i}$ as $\chi_{j,i,1} \circ \chi_{j,i,2} \circ \cdots \circ \chi_{j,i,\kappa}$. Namely, 1, 3, 5 $(\widetilde{\phi})$ follow at once from the corresponding conditions (χ). Condition 4 $(\widetilde{\phi})$ follows from 4 (χ) and 3 (χ), since by 3 (χ) (1) and 3 (χ) (2) the supports of the homeomorphism $(\overline{\chi}_{j,i,k})_1$ and the isotopy $\chi_{j,i,k+1}$ are disjoint for $1 \le k \le \kappa - 1$. Thus $(\widetilde{\phi}_{j,i})_t$ moves each point, for each t, by no more than $\delta_{j,i,k}$, which by 4 (χ) is independent of k. Condition 2 $(\widetilde{\phi})$ follows from 2 (χ), taken for all k, $1 \le k \le \kappa$, since the supports of the homeomorphisms $(\overline{\chi}_{j,i,k})_1$ are disjoint by 3 (χ) (2).

4.20. As we have already observed, it follows from conditions 3 (χ) that the isotopies $\overline{\chi}_{j,i,k}$ are so constructed that $S(\overline{\chi}_{j,i,k}) \cap S((\overline{\chi}_{j,i,k-1})_1) = \Lambda$ for $k > 1$. Therefore we can construct these isotopies independently of one another. We effect the construction of the isotopies $\overline{\chi}_{j,i,k}$ on the basis of the lemma on the correction of homeomorphisms.

We apply the following case of this lemma:

The axis ox_i is fixed for a given value of i, the numbers r_0, $\overline{r}$, and r_1 of the lemma are our respective numbers r_{i-1}, $\overline{r}$, r_i, for $i' \ne i$ the numbers $d_{i',k}$ of the lemma are the numbers of order $(i-1, i')$, and d_0, d_1, d_2, d_3 are $d_{k-2}, d_{k-1}, d_k, d_{k+1}$ respectively. Finally, the c_l of the lemma are the numbers of order (i, i) interior to $[d_{k-1}, d_k]$. If $\epsilon = \epsilon_{j-1}$, then the condition that the ϵ-zones of the cubes $I_{i'}(d; r_0)$ *referred to in the lemma are disjoint is satisfied, as was shown above in* 4.14.

Now let the homeomorphism g of the lemma be $\overline{g}_{i-1}$. Then by 2 $(\widetilde{g})$ it satisfies condition $(*)$, and so we may construct an isotopy $X(\overline{g}_{i-1})$ with the properties 1–3 (X). It is easy to verify that $X(\overline{g}_{i-1})$ can be taken as the required isotopy $\chi_{j,i,k}$.

Indeed, conditions 1–3 (χ) follow at once: 1 (χ) from 1 (X) and 1 $(\widetilde{g})$, and 2 (χ), 3 (χ) from

from 2 (X), 3 (X) respectively. As for 4 (χ), it follows from the estimates given in 3.19. For by condition 4 (X) there, X ($\bar{g}_{i-1}$) is an $f(n, \lambda)$ $(\eta + 2\epsilon)$-isotopy, where $f(n, \lambda)$ depends only on n and λ. But, as is easily computed, λ is equal to $2 \cdot 3^{n-1}$ or $3^n - 1$ according as $[d_{k-1}, d_k]$ lies in

$$\left[-1-\frac{1}{2^{j+1}},\ 1+\frac{1}{2^{j+1}}\right]$$

or in one of the intervals

$$\left[-r_i,\ -1-\frac{1}{2^{j+1}}\right] \text{ or } \left[1+\frac{1}{2^{j+1}},\ r_i\right].$$

In either case λ is less than 3^n, and so $f(n, \lambda)$ is less than some number depending only on n, and in particular independent of k.

On the other hand, η and ϵ are less than $1/2^{j-1}$, and therefore we obtain that X$(\bar{g}_{i-1})$ is an $\bar{f}(n)/2^j$-isotopy, where $\bar{f}(n)$ depends only on n.

Finally, condition 5 (χ) follows from 5 (X) (see 3.19) in the case when $d_k \neq \pm 1$, while if $d_k = \pm 1$ we alter the construction of X in accordance with the Remark in 3.20.

Thus all the conditions 1–5 (χ) are satisfied, and the proof is complete.

§5. Corollaries and remarks

5.1. We first remak that Proposition (B) can be strengthened in various ways. For example, it can be stipulated that the isotopies H(g) are constructed for all homeomorphisms subordinate to some majorant f on O, in such a way that when the homeomorphism is the identity on D the isotopy is the identity on $D \setminus O'$. Also, not just one but any finite number of such sets may be taken. Further, as is easily seen from the proof, if D is a locally flat submanifold of M then in the statement of (B) the neighborhood O' can be omitted; that is, an isotopy can be constructed which is the identity on the whole of D. We do not know whether this can be done if D is an arbitrary polyhedron (even one with handles) or, indeed, an ar bitrary compact set.

5.2. Some applications to stable manifolds (see [13]) can be made simply on the basis of the local linear connectedness of $\mathfrak{H}_m(M)$. We note that the isotopies constructed in the proof of (B) do not take a homeomorphism out of its stable class. Hence we obviously deduce

Corollary 1. *The stable classes in the group $\mathfrak{H}_m(M)$ coincide with the components of linear connectedness of the group $\mathfrak{H}_m(M)$.*

Corollary 2. *If a homeomorphism can be approximated by stable ones then it is itself stable* (*cf.* [14]).

5.3. We further note the

Corollary 3. *If D is a closed subset of the manifold M, and D' a closed subset of M lying in* Int D, *then there is a neighborhood $\Omega(e)$ in $\mathfrak{H}_m(M)$ such that for every homeomorphism h in this neighborhood a homeomorphism of the manifold can be found which depends continuously on h and is equal to h on D' and to e on $M \setminus D$.*

From this it can easily be deduced that the covering homotopy axiom is satisfied in the space of imbeddings of manifolds with normal microfibrations. However we shall not enter into this in detail here, since by means of a certain modification of the present method we shall, in a later paper, extend these results to the space of locally flat imbeddings, from which, in particular, the covering homotopy theorem will also follow for the space of these imbeddings, represented as a factor space of the group of homeomorphisms of the enveloping manifold.

5.4. While the present manuscript was in course of publication, it became known that two American mathematicians—Kirby and Siebenmann—had constructed piecewise linear homeomorphisms of a multi-dimensional torus which were arbitrarily close to the identity, but not piecewise linear isotopic to the identity. Thus the main result of this paper does not carry over to piecewise linear (nor, consequently, smooth) homeomorphisms. Moreover, using the fact that, by the theorem we have proved here, their homeomorphisms are isotopic to the identity in the topological sense, Kirby and Siebenmann deduced that the known obstructions to the existence and uniqueness of piecewise linear structures taking values in the group $\pi_3(\mathrm{Top}/pl)$ can be nontrivial, and so this group is Z_2 (and not null). This implies the existence of combinatorially nontriangulable and also of combinatorially nonequivalent piecewise linear structures on certain manifolds (in particular, on tori). We remark that this also implies the existence of nonapproximable piecewise linear locally flat imbeddings of piecewise linear manifolds of codimension unity. For, as we stated in the preceding subsection, by the method of this paper it can be shown that sufficiently close locally flat imbeddings (of codimension different from 2) are isotopic. Thanks to this fact, it may be easy, using the piecewise linear approximability of imbeddings of codimension unity, to prove the triangulability of an arbitrary topological manifold by induction on the number of coordinate neighborhoods of the manifold.

Further, it is not hard to show that imbeddings of manifolds of the form $S^p \times S^q$ in R^n, where $p + q = n - 1$, can already be piecewise linear nonapproximable.

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